# Perturbation Theory of Coulomb Gauge Yang-Mills Theory Within the First Order Formalism

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Perturbative Coulomb gauge Yang-Mills theory within the first order formalism is considered. Using a differential equation technique and dimensional regularization, analytic results for both the ultraviolet divergent and finite parts of the two-point functions at one-loop order are derived. It is shown how the non-ultraviolet divergent parts of the results are finite at spacelike momenta with kinematical singularities on the light-cone and subsequent branch cuts extending into the timelike region.

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## 1. INTRODUCTION

Coulomb gauge QCD is rather special. Amongst all the various gauges, it can be shown in Coulomb gauge that the number of dynamical variables reduces to the number of physical degrees of freedom [1]. This allows for a tantalizing glimpse of possible nonperturbative descriptions of confinement and the hadron spectrum. The so-called "Gribov–Zwanziger" scenario of confinement, [1, 2], becomes especially relevant in Coulomb gauge. In this picture, the temporal component of the gluon propagator provides for a long-range confining force whilst the transverse spatial components are suppressed in the infrared. Various calculations support this picture, among them [3, 4, 5, 6, 7].

Perhaps a touch ironically, perturbation theory is one of the starting ingredients for nonperturbative calculations in the sense that in the high energy region (and for asymptotically free theories such as QCD this is the perturbative domain), the regularization and renormalization of the theory play an important role in constraining the necessary approximations. In Coulomb gauge, only the leading divergence structure is known due to severe technical difficulties (see for example [8, 9]). For Coulomb gauge within the (standard) second order formalism there exist so-called energy divergences, which have been shown to cancel up to two-loops [9], but a general proof of this cancellation is sadly lacking. One way to circumvent the energy divergences is to work within the first order formalism, where formal arguments show that such divergences cancel exactly [1]. This circumvention comes at a price: the Dyson–Schwinger equations become cumbersome [10] and full multiplicative renormalizability is not maintained [1]. Whilst the leading divergences do provide crucial information about the renormalization of the theory, the remaining finite parts are critical to further progress in the field. Having expounded the necessity for perturbative results within the nonperturbative context, obviously perturbative results for physical high energy processes are desirable – not the least in order to compare with results from covariant gauges.

The technical barrier to progress in Coulomb gauge perturbation theory stems from noncovariant loop integrals of the type:

$$\int \frac{d^4\omega}{\omega^2(k-\omega)^2\vec{\omega}^2(\vec{k}-\vec{\omega})^2} \tag{1.1}$$

where (in Euclidean space) $\omega^2 = \omega_4^2 + \vec{\omega}^2$. Standard techniques such as Schwinger parametrization [11] fail due to the complexity of the resulting parametric integrals. One might imagine that using contour integration to firstly evaluate the temporal component of the integral might make the situation simpler. However, in such a method, translational invariance is lost and since the subsequent spatial integral is ultraviolet [UV] divergent, the result will in general be incorrect. A UV-cutoff procedure will also fail. There are however techniques that can overcome these difficulties and one of these is the differential equation technique. In its original form [12, 13], complicated massive integrals arising in covariant gauge calculations can be considered and when supplemented with integration by parts identities [14, 15], the technique becomes a powerful tool. The ethos of the technique is that whilst the multi–dimensional parametric form of the integral may be practically impossible to work with, the original integral is itself only a function of a few variables and where differential equations can be derived, finding the solution involves integration over only these few variables and sorting out the boundary conditions.

In this paper, we consider the one-loop perturbative two-point functions of Coulomb gauge Yang-Mills theory within the first order formalism. Within this noncovariant setting, a variant of the differential equation technique and the integration by parts identities are derived in order to evaluate integrals such as the one above. This allows for a full analysis of the various propagator and two-point proper functions of the theory.

The paper is organized as follows. We start by briefly reviewing the first order formalism used and express the two-point functions in terms of their loop integrals. In Section 3, the noncovariant integrals inherent to Coulomb gauge are evaluated. This section comprises the bulk of the development necessary to the study and is unashamedly technical in nature. The results for the two-point functions are collected in Section 4. We finish with a summary and outlook. Those loop integrals that can be evaluated using standard techniques are described in Appendix A. Appendix B contains a nontrivial check on the noncovariant integrals.

## 2. THE FIRST ORDER FORMALISM AND PERTURBATION THEORY

Since Coulomb gauge Yang-Mills theory within the first order formalism is rather different to Yang-Mills theory in linear covariant gauges, let us begin by reviewing those aspects of the formalism that will be relevant. For a complete description, the reader is referred to Ref. [10]. The generating functional is written (in Minkowski space)

$$Z[J] = \int \mathcal{D}\Phi \exp\left\{\imath\mathcal{S}_B + \imath\mathcal{S}_{fp} + \imath\mathcal{S}_\pi + \imath\mathcal{S}_s\right\} \tag{2.1}$$

where $\Phi$ denotes the collection of fields and the terms in the action are given by

$$\begin{aligned}
\mathcal{S}_B &= \int d^4x \left[-\frac{1}{2}\vec{B}^a\cdot\vec{B}^a\right], \\
\mathcal{S}_{fp} &= \int d^4x \left[-\lambda^a\vec{\nabla}\cdot\vec{A}^a - \overline{c}^a\vec{\nabla}\cdot\vec{D}^{ab}c^b\right], \\
\mathcal{S}_\pi &= \int d^4x \left[-\tau^a\vec{\nabla}\cdot\vec{\pi}^a - \frac{1}{2}(\vec{\pi}^a - \vec{\nabla}\phi^a)\cdot(\vec{\pi}^a - \vec{\nabla}\phi^a) + (\vec{\pi}^a - \vec{\nabla}\phi^a)\cdot\left(\partial^0\vec{A}^a + \vec{D}^{ab}\sigma^b\right)\right]
\end{aligned} \tag{2.2}$$

with the source term defined in condensed notation as (Greek indices such as $\alpha$ refer to all attributes of the field, including its type, and summation over all discrete indices and integration over all continuous arguments is implicitly understood):

$$\mathcal{S}_s = J_\alpha\Phi_\alpha. \tag{2.3}$$

In the above, $\vec{A}$ and $\sigma$ are the spatial and temporal components of the gauge field, $\vec{\pi}$ and $\phi$ arise in the construction of the first order formalism (they represent the transverse and longitudinal components of the conjugate momentum to the gauge field), $\overline{c}$ and $c$ are the Grassmann-valued Faddeev–Popov ghost fields introduced by fixing the gauge, $\lambda$ and $\tau$ are Lagrange multiplier fields. The chromomagnetic field, $\vec{B}$, is given by

$$B_i^a = \epsilon_{ijk}\left[\nabla_j A_k^a - \frac{1}{2}gf^{abc}A_j^bA_k^c\right] \tag{2.4}$$

(roman subscripts indicate spatial indices) and the spatial component of the covariant derivative in the adjoint representation is

$$\vec{D}^{ab} = \delta^{ab}\vec{\nabla} - gf^{acb}\vec{A}^c. \tag{2.5}$$

The general forms of the various Green's functions that we will be considering are constrained in several ways. Since the derivation is necessarily somewhat longwinded, for brevity we omit the details here and again refer the reader to Ref. [10] for a full account. There are three constraints. Firstly, the Lagrange multiplier ($\lambda$, $\tau$) and $\phi$ field equations of motion can be solved exactly. The former primarily supply the transversality properties of the vector–vector propagators (i.e., the connected two-point Green's functions), the latter relating the proper two-point functions involving functional derivatives of the $\phi$-field to contractions of those involving the corresponding $\vec{\pi}$-field. Secondly, the equation stemming from the invariance of the generating functional under the BRS transform (the Ward–Takahashi identity in raw functional form) tells us that the $\lambda$–$\lambda$ propagator must vanish. Thirdly, the discrete symmetries of time-reversal and parity constrain the allowed forms with two main consequences: most of the scalar–vector propagators must vanish (this is applied in conjunction with the transversality conditions arising from enforcing the Lagrange multiplier equations of motion) and the dressing functions of the propagator or two-point proper Green's functions must be functions of the variables $k_0^2$ and $\vec{k}^2$.

In Table I, the general decomposition of the propagators (collectively denoted by $W$) is presented. The vector–vector propagators are explicitly transverse, with the transverse projector in momentum space given by $t_{ij}(\vec{k}) = \delta_{ij} - k_ik_j/\vec{k}^2$.

It is understood that the denominator factors involving both temporal and spatial components implicitly carry the appropriate Feynman prescription, i.e.,

$$\frac{1}{\left(k_0^2-\vec{k}^2\right)} \to \frac{1}{\left(k_0^2-\vec{k}^2+\imath 0_+\right)}, \tag{2.6}$$

such that the integral over the temporal component can be analytically continued to Euclidean space ($k_0 \to \imath k_4$). Supplemented by the additional expression for the ghost propagator,

$$W_{\bar{c}c}^{ab}(k) = -\delta^{ab}\frac{\imath D_c}{\vec{k}^2}, \tag{2.7}$$

the list is complete. Each of the dressing functions $D_{\alpha\beta}$ is a function of $k_0^2$ and $\vec{k}^2$ except the ghost which is a function of $\vec{k}^2$ only. The tree-level propagators are given by

$$D_{AA} = D_{A\pi} = D_{\pi\pi} = D_{\sigma\sigma} = D_{\sigma\phi} = D_{\sigma\lambda} = D_c = 1, \qquad D_{\phi\phi} = D_{\phi\lambda} = 0. \tag{2.8}$$

The general decomposition of the proper two-point functions (collectively denoted by $\Gamma$) is given in Table II. The vector–vector functions contain longitudinal components and the longitudinal projector is written $l_{ij}(\vec{k}) = k_i k_j/\vec{k}^2$. The ghost proper two-point function is:

$$\Gamma_{\bar{c}c}^{ab}(k) = \delta^{ab}\imath\Gamma_c\vec{k}^2. \tag{2.9}$$

Again, the dressing functions are functions of $k_0^2$ and $\vec{k}^2$ except that for the ghost which is a function of $\vec{k}^2$ only. At tree-level

$$\Gamma_{AA} = \Gamma_{A\pi} = \Gamma_{\pi\pi} = \Gamma_{\pi\sigma} = \Gamma_c = 1, \qquad \overline{\Gamma}_{AA} = \overline{\Gamma}_{A\pi} = \overline{\Gamma}_{\pi\pi} = \Gamma_{A\sigma} = \Gamma_{\sigma\sigma} = 0. \tag{2.10}$$

The two sets of functions (propagator and two-point proper Green's functions) are related via the Legendre transform and we have

$$\begin{aligned}
D_{AA} &= \frac{\left(k_0^2-\vec{k}^2\right)\Gamma_{\pi\pi}}{\left(k_0^2\Gamma_{A\pi}^2-\vec{k}^2\Gamma_{AA}\Gamma_{\pi\pi}\right)}, & D_{\sigma\sigma} &= \frac{\left(\Gamma_{\pi\pi}+\overline{\Gamma}_{\pi\pi}\right)}{\Gamma_{\pi\sigma}^2-\Gamma_{\sigma\sigma}\left(\Gamma_{\pi\pi}+\overline{\Gamma}_{\pi\pi}\right)},\\
D_{\pi\pi} &= \frac{\left(k_0^2-\vec{k}^2\right)\Gamma_{AA}}{\left(k_0^2\Gamma_{A\pi}^2-\vec{k}^2\Gamma_{AA}\Gamma_{\pi\pi}\right)}, & D_{\phi\phi} &= -\frac{\Gamma_{\sigma\sigma}}{\Gamma_{\pi\sigma}^2-\Gamma_{\sigma\sigma}\left(\Gamma_{\pi\pi}+\overline{\Gamma}_{\pi\pi}\right)},\\
D_{A\pi} &= \frac{\left(k_0^2-\vec{k}^2\right)\Gamma_{A\pi}}{\left(k_0^2\Gamma_{A\pi}^2-\vec{k}^2\Gamma_{AA}\Gamma_{\pi\pi}\right)}, & D_{\sigma\phi} &= \frac{\Gamma_{\pi\sigma}}{\Gamma_{\pi\sigma}^2-\Gamma_{\sigma\sigma}\left(\Gamma_{\pi\pi}+\overline{\Gamma}_{\pi\pi}\right)},\\
D_c\Gamma_c &= 1, & &\\
0 &= D_{\sigma\sigma}\Gamma_{A\sigma} - D_{\sigma\phi}\left(\Gamma_{A\pi}+\overline{\Gamma}_{A\pi}\right) + D_{\sigma\lambda}, & &\\
0 &= -D_{\sigma\phi}\Gamma_{A\sigma} - D_{\phi\phi}\left(\Gamma_{A\pi}+\overline{\Gamma}_{A\pi}\right) + D_{\phi\lambda}, & &\\
0 &= \overline{\Gamma}_{AA} - \frac{k_0^2}{\vec{k}^2}\left[D_{\sigma\lambda}\Gamma_{A\sigma} + D_{\phi\lambda}\left(\Gamma_{A\pi}+\overline{\Gamma}_{A\pi}\right)\right]. & &
\end{aligned} \tag{2.11}$$

In addition to the two-point functions we have various three-point functions. The tree-level vertices (three-point proper Green's functions) used in this work are given by (all momenta are defined as incoming):

$$\begin{aligned}
\Gamma_{\pi\sigma Aij}^{(0)abc} &= -gf^{abc}\delta_{ij},\\
\Gamma_{3Aijk}^{(0)abc}(p_a,p_b,p_c) &= -\imath gf^{abc}\left[\delta_{ij}(p_a-p_b)_k + \delta_{jk}(p_b-p_c)_i + \delta_{ki}(p_c-p_a)_j\right],\\
\Gamma_{4Aijkl}^{(0)abcd} &= -\imath g^2\left\{\delta_{ij}\delta_{kl}\left[f^{ace}f^{bde}-f^{ade}f^{cbe}\right] + \delta_{ik}\delta_{jl}\left[f^{abe}f^{cde}-f^{ade}f^{bce}\right] + \delta_{il}\delta_{jk}\left[f^{ace}f^{dbe}f^{abe}f^{cde}\right]\right\},\\
\Gamma_{\bar{c}cAi}^{(0)abc}(p_{\bar{c}},p_c,p_A) &= -\imath gf^{abc}p_{\bar{c}i},\\
\Gamma_{\phi\sigma Ai}^{(0)abc}(p_\phi,p_\sigma,p_A) &= \imath p_{\phi j}\Gamma_{\pi\sigma Aji}^{(0)abc} = -\imath gf^{abc}p_{\phi i}.
\end{aligned} \tag{2.12}$$

The (Minkowski space) loop integration measure for the integrals entering the Dyson–Schwinger equations is $(-\,\text{đ}\omega_M)$ where

$$\text{đ}\omega_M = \frac{d\omega_0\, d^d\vec{\omega}}{(2\pi)^{d+1}} \tag{2.13}$$

| $W$ | $A_j$ | $\pi_j$ | $\sigma$ | $\phi$ | $\lambda$ | $\tau$ |
|---|---|---|---|---|---|---|
| $A_i$ | $t_{ij}(\vec{k})\frac{\imath D_{AA}}{(k_0^2-\vec{k}^2)}$ | $t_{ij}(\vec{k})\frac{(-k^0)D_{A\pi}}{(k_0^2-\vec{k}^2)}$ | 0 | 0 | $\frac{(-k_i)}{\vec{k}^2}$ | 0 |
| $\pi_i$ | $t_{ij}(\vec{k})\frac{k^0 D_{A\pi}}{(k_0^2-\vec{k}^2)}$ | $t_{ij}(\vec{k})\frac{\imath\vec{k}^2 D_{\pi\pi}}{(k_0^2-\vec{k}^2)}$ | 0 | 0 | 0 | $\frac{(-k_i)}{\vec{k}^2}$ |
| $\sigma$ | 0 | 0 | $\frac{\imath D_{\sigma\sigma}}{\vec{k}^2}$ | $\frac{-\imath D_{\sigma\phi}}{\vec{k}^2}$ | $\frac{(-k^0)D_{\sigma\lambda}}{\vec{k}^2}$ | 0 |
| $\phi$ | 0 | 0 | $\frac{-\imath D_{\sigma\phi}}{\vec{k}^2}$ | $\frac{-\imath D_{\phi\phi}}{\vec{k}^2}$ | $\frac{(-k^0)D_{\phi\lambda}}{\vec{k}^2}$ | $\frac{\imath}{\vec{k}^2}$ |
| $\lambda$ | $\frac{k_j}{\vec{k}^2}$ | 0 | $\frac{k^0 D_{\sigma\lambda}}{\vec{k}^2}$ | $\frac{k^0 D_{\phi\lambda}}{\vec{k}^2}$ | 0 | 0 |
| $\tau$ | 0 | $\frac{k_j}{\vec{k}^2}$ | 0 | $\frac{\imath}{\vec{k}^2}$ | 0 | 0 |

TABLE I: General form of propagators in momentum space. The global color factor $\delta^{ab}$ has been extracted. All unknown functions $D_{\alpha\beta}$ are dimensionless, scalar functions of $k_0^2$ and $\vec{k}^2$.

| $\Gamma$ | $A_j$ | $\pi_j$ | $\sigma$ | $\lambda$ | $\tau$ |
|---|---|---|---|---|---|
| $A_i$ | $t_{ij}(\vec{k})\imath\vec{k}^2\Gamma_{AA}+\imath k_ik_j\overline{\Gamma}_{AA}$ | $k^0\left(\delta_{ij}\Gamma_{A\pi}+l_{ij}(\vec{k})\overline{\Gamma}_{A\pi}\right)$ | $-\imath k^0k_i\Gamma_{A\sigma}$ | $k_i$ | 0 |
| $\pi_i$ | $-k^0\left(\delta_{ij}\Gamma_{A\pi}+l_{ij}(\vec{k})\overline{\Gamma}_{A\pi}\right)$ | $\imath\delta_{ij}\Gamma_{\pi\pi}+\imath l_{ij}(\vec{k})\overline{\Gamma}_{\pi\pi}$ | $k_i\Gamma_{\pi\sigma}$ | 0 | $k_i$ |
| $\sigma$ | $-\imath k^0k_j\Gamma_{A\sigma}$ | $-k_j\Gamma_{\pi\sigma}$ | $\imath\vec{k}^2\Gamma_{\sigma\sigma}$ | 0 | 0 |
| $\lambda$ | $-k_j$ | 0 | 0 | 0 | 0 |
| $\tau$ | 0 | $-k_j$ | 0 | 0 | 0 |

TABLE II: General form of the proper two-point functions in momentum space. The global color factor $\delta^{ab}$ has been extracted. All unknown functions $\Gamma_{\alpha\beta}$ are dimensionless, scalar functions of $k_0^2$ and $\vec{k}^2$.

and $d=3-2\varepsilon$ is the spatial dimension. In order to preserve the dimension of the dressing functions (or, more formally, the action) we must also assign a dimension to the coupling through the replacement

$$g^2\rightarrow g^2\mu^\varepsilon \tag{2.14}$$

where $\mu$ is the square of some non-vanishing mass scale (and which will be later identified with the renormalization scale).

Although the formalism as presented thus far is written in Minkowski space, in order to evaluate the resulting loop integrals it is necessary to analytically continue to Euclidean space ($k_0\rightarrow\imath k_4$) and we use the notation $k_4$ to denote the temporal component of the Euclidean 4-momentum such that $k^2=k_4^2+\vec{k}^2$. The Euclidean integration measure is written

$$\,{\mathchar'26\mkern-12mu d}\omega=\frac{d\omega_4\,d^d\vec{\omega}}{(2\pi)^{d+1}}. \tag{2.15}$$

Since, in this noncovariant setting, the evaluated dressing functions will be functions of the two variables $k_4^2$ and $\vec{k}^2$, the Minkowski space dressing functions are recovered by analytically continuing back with $k_4^2\rightarrow-k_0^2$. Assuming that the loop integrals can be expressed in terms of known analytic functions, the Minkowski space dressing functions can be given by simply extending the argument $k_4^2$ to negative values and observing contributions generated by continuing through singularities. Physically, such singularities can only occur for lightlike momenta (assuming that there are no timelike resonance states) just as in linear covariant gauges. We will use the same notation for the Minkowski and Euclidean space functions, since it is clear from the argument $k_0^2$ or $k_4^2$ in which space they reside.

Now, the stated purpose of this paper is to derive the one-loop perturbative form for the various two-point dressing functions. In linear covariant gauges, the Feynman rules can be easily applied to give directly the expressions for propagator functions. The reason for the relative simplicity is that the proper two-point functions are related directly to the corresponding propagators via inversion. This is not so within the first order formalism – since the proper

two-point Green's functions and the propagators are related effectively by a matrix inversion, one must expand the full set of Dyson–Schwinger equations for the proper functions in powers of the coupling, $g$, and then use the relations (2.11) to construct the propagators. [Actually, a similar situation exists for the quark propagator in covariant gauges too, but the matrix inversion only involves different contractions of the same equation.] In the remainder of this section, we will thus consider the one-loop perturbative expansion of the various Dyson–Schwinger equations and write the dressing functions of the proper two-point functions in terms of the loop integrals (these integrals will be evaluated in the course of the next section). There are seven such equations (derived in [10], with the exception of $\Gamma_{\sigma A}$ which is an obvious extension of $\Gamma_{\sigma\sigma}$) and they read:

$$\Gamma_{\pi Aik}^{ad}(k) = -\delta^{ad}k^0\delta_{ik} - \int(-đ\omega_M)\Gamma_{\pi\sigma Aij}^{(0)abc}(k,-\omega,\omega-k)W_{\sigma\beta}^{be}(\omega)\Gamma_{\beta\alpha Alk}^{efd}(\omega,k-\omega,-k)W_{\alpha Alj}^{fc}(\omega-k), \tag{2.16}$$

$$\Gamma_{c}^{ad}(k) = \delta^{ad}\imath\vec{k}^2 - \int(-đ\omega_M)\Gamma_{\bar{c}cAi}^{(0)abc}(k,-\omega,\omega-k)W_{c}^{be}(\omega)W_{A\alpha ij}^{cf}(k-\omega)\Gamma_{\bar{c}c\alpha j}^{edf}(\omega,-k,k-\omega), \tag{2.17}$$

$$\Gamma_{\pi\sigma i}^{ad}(k) = \delta^{ad}k_i - \int(-đ\omega_M)\Gamma_{\pi\sigma Aij}^{(0)abc}(k,-\omega,\omega-k)W_{\sigma\beta}^{be}(\omega)\Gamma_{\beta\alpha\sigma l}^{efd}(\omega,k-\omega,-k)W_{\alpha Alj}^{fc}(\omega-k), \tag{2.18}$$

$$\Gamma_{\pi\pi ik}^{ad}(k) = \imath\delta^{ad}\delta_{ik} - \int(-đ\omega_M)\Gamma_{\pi\sigma Aij}^{(0)abc}(k,-\omega,\omega-k)W_{\sigma\beta}^{be}(\omega)\Gamma_{\beta\alpha\pi lk}^{efd}(\omega,k-\omega,-k)W_{\alpha Alj}^{fc}(\omega-k), \tag{2.19}$$

$$\begin{aligned}\Gamma_{\sigma Am}^{ad}(k) = &-\int(-đ\omega_M)\Gamma_{\pi\sigma Aij}^{(0)cab}(\omega-k,k,-\omega)W_{A\beta jl}^{be}(\omega)\Gamma_{\beta\alpha Alkm}^{efd}(\omega,k-\omega,-k)W_{\alpha\pi ki}^{fc}(\omega-k)\\ &-\int(-đ\omega_M)\Gamma_{\phi\sigma Ai}^{(0)cab}(\omega-k,k,-\omega)W_{A\beta ij}^{be}(\omega)\Gamma_{\beta\alpha Ajm}^{efd}(\omega,k-\omega,-k)W_{\alpha\phi}^{fc}(\omega-k),\end{aligned} \tag{2.20}$$

$$\begin{aligned}\Gamma_{\sigma\sigma}^{ad}(k) = &-\int(-đ\omega_M)\Gamma_{\pi\sigma Aij}^{(0)cab}(\omega-k,k,-\omega)W_{A\beta jl}^{be}(\omega)\Gamma_{\beta\alpha\sigma lk}^{efd}(\omega,k-\omega,-k)W_{\alpha\pi ki}^{fc}(\omega-k)\\ &-\int(-đ\omega_M)\Gamma_{\phi\sigma Ai}^{(0)cab}(\omega-k,k,-\omega)W_{A\beta ij}^{be}(\omega)\Gamma_{\beta\alpha\sigma j}^{efd}(\omega,k-\omega,-k)W_{\alpha\phi}^{fc}(\omega-k),\end{aligned} \tag{2.21}$$

$$\begin{aligned}\Gamma_{AAim}^{ae}(k) = &\ \imath\delta^{ae}\left[\vec{k}^2\delta_{im}-k_ik_m\right] + \int(-đ\omega_M)\Gamma_{\bar{c}cAi}^{(0)bca}(\omega-k,-\omega,k)W_{c}^{cd}(\omega)\Gamma_{\bar{c}cAm}^{dfe}(\omega,k-\omega,-k)W_{c}^{fb}(\omega-k)\\ &-\int(-đ\omega_M)\Gamma_{\phi\sigma Ai}^{(0)bca}(\omega-k,-\omega,k)W_{\sigma\beta}^{cd}(\omega)\Gamma_{\beta\alpha Am}^{dfe}(\omega,k-\omega,-k)W_{\alpha\phi}^{fb}(\omega-k)\\ &-\int(-đ\omega_M)\Gamma_{\pi\sigma Aij}^{(0)bca}(\omega-k,-\omega,k)W_{\sigma\beta}^{cd}(\omega)\Gamma_{\beta\alpha Akm}^{dfe}(\omega,k-\omega,-k)W_{\alpha\pi kj}^{fb}(\omega-k)\\ &-\frac{1}{2}\int(-đ\omega_M)\Gamma_{3Akji}^{(0)bca}(\omega-k,-\omega,k)W_{A\beta jl}^{cd}(\omega)\Gamma_{\beta\alpha Alnm}^{dfe}(\omega,k-\omega,-k)W_{\alpha Ank}^{fb}(\omega-k)\\ &-\frac{1}{6}\int(-đ\omega_M)(-đ v_M)\Gamma_{4Alkji}^{(0)dcba}(-v,-\omega,v+\omega-k,k)W_{A\lambda jn}^{bf}(k-v-\omega)W_{A\gamma ko}^{cg}(\omega)W_{A\delta lp}^{dh}(v)\times\\ &\Gamma_{\lambda\gamma\delta Anopm}^{fghe}(k-\omega-v,\omega,v,-k)\\ &+\frac{1}{2}\int(-đ\omega_M)\Gamma_{4Aimlk}^{(0)aecd}(k,-k,\omega,-\omega)W_{AAkl}^{cd}(-\omega)\\ &+\frac{1}{2}\int(-đ\omega_M)(-đ v_M)\Gamma_{4Alkji}^{(0)dcba}(-v,-\omega,v+\omega-k,k)W_{A\delta ln}^{df}(v)W_{A\gamma ko}^{cg}(\omega)\Gamma_{\delta\gamma\lambda nop}^{fgh}(v,\omega,-v-\omega)\times\\ &W_{\lambda\mu pq}^{hi}(v+\omega)\Gamma_{\mu\nu Aqrm}^{ije}(v+\omega,k-v-\omega,-k)W_{\nu Arj}^{jd}(\omega+v-k).\end{aligned} \tag{2.22}$$

These equations are all expanded to one-loop, the contributing terms tabulated in Table III. Introducing some notation for the tree-level and one-loop terms, we write

$$\Gamma_{\alpha\beta} = \Gamma_{\alpha\beta}^{(0)} + g^2\Gamma_{\alpha\beta}^{(1)} \tag{2.23}$$

where the dimensionful parameter $\mu^{\varepsilon}$ is included in $\Gamma^{(1)}$. After inserting the appropriate tree-level factors, resolving the color algebra and analytically continuing to Euclidean space, the one-loop expressions read:

$$\delta_{ij}\Gamma_{A\pi}^{(1)}(k_4^2,\vec{k}^2) + l_{ij}(\vec{k})\overline{\Gamma}_{A\pi}^{(1)}(k_4^2,\vec{k}^2) = -N_c\mu^{\varepsilon}\int\frac{đ\omega\ \omega_4}{k_4\omega^2(\vec{k}-\vec{\omega})^2}\left[\delta_{ik}-\frac{\omega_i\omega_k}{\vec{\omega}^2}\right], \tag{2.24}$$

$$\Gamma_{c}^{(1)}(\vec{k}^2) = -N_c\mu^{\varepsilon}\int\frac{đ\omega}{\omega^2(\vec{k}-\vec{\omega})^2}\left[1-\frac{\vec{k}\cdot\vec{\omega}^2}{\vec{k}^2\vec{\omega}^2}\right], \tag{2.25}$$

| equation | integral term | contribution(s) $(\alpha, \beta, \ldots)$ |
|---|---|---|
| $\Gamma_{\pi A}$, (2.16) | 1st | $(\alpha = \pi, \beta = \sigma)$ |
| $\Gamma_c$, (2.17) | 1st | $(\alpha = A)$ |
| $\Gamma_{\pi\sigma}$, (2.18) | 1st | $(\alpha = A, \beta = \phi)$ |
| $\Gamma_{\pi\pi}$, (2.19) | 1st | $(\alpha = A, \beta = \sigma)$ |
| $\Gamma_{\sigma A}$, (2.20) | 1st | $(\alpha = A, \beta = A)$ |
| | 2nd | $(\alpha = \sigma, \beta = \pi)$ |
| $\Gamma_{\sigma\sigma}$, (2.21) | 1st | $(\alpha = A, \beta = \pi)$,$(\alpha = \pi, \beta = A)$ |
| | 2nd | — |
| | 1st | (explicit) |
| | 2nd | $(\alpha = \sigma, \beta = \phi)$ |
| | 3rd | $(\alpha = \pi, \beta = \sigma)$ |
| $\Gamma_{AA}$, (2.22) | 4th | $(\alpha = A, \beta = A)$ |
| | 5th | — |
| | 6th | — |
| | 7th | — |

TABLE III: Contributing integral terms to the one-loop perturbative expressions for the Dyson–Schwinger equations (2.16–2.22).

$$\Gamma^{(1)}_{\pi\sigma}(k_4^2,\vec{k}^2) = -N_c\mu^\varepsilon \int \frac{\text{\dj}\omega}{\omega^2(\vec{k}-\vec{\omega})^2}\left[1-\frac{\vec{k}\cdot\vec{\omega}^2}{\vec{k}^2\vec{\omega}^2}\right], \tag{2.26}$$

$$\delta_{ik}\Gamma^{(1)}_{\pi\pi}(k_4^2,\vec{k}^2) + l_{ik}(\vec{k})\overline{\Gamma}^{(1)}_{\pi\pi}(k_4^2,\vec{k}^2) = N_c\mu^\varepsilon \int \frac{\text{\dj}\omega}{\omega^2(\vec{k}-\vec{\omega})^2}\left[\delta_{ik}-\frac{\omega_i\omega_k}{\vec{\omega}^2}\right], \tag{2.27}$$

$$\begin{aligned}\Gamma^{(1)}_{A\sigma}(k_4^2,\vec{k}^2) &= N_c\mu^\varepsilon \int \frac{\text{\dj}\omega\ \omega_4\vec{k}\cdot(\vec{k}-2\vec{\omega})}{k_4\vec{k}^2\omega^2(k-\omega)^2}\left[d-1-\frac{\vec{k}^2}{\vec{\omega}^2}+\frac{\vec{k}\cdot(\vec{k}-\vec{\omega})^2}{\vec{\omega}^2(\vec{k}-\vec{\omega})^2}\right] \\ &\quad + N_c\mu^\varepsilon \int \frac{\text{\dj}\omega\ w_4}{k_4\omega^2(\vec{k}-\vec{\omega})^2}\left[1-\frac{\vec{k}\cdot\vec{\omega}^2}{\vec{k}^2\vec{\omega}^2}\right],\end{aligned} \tag{2.28}$$

$$\Gamma^{(1)}_{\sigma\sigma}(k_4^2,\vec{k}^2) = -N_c\mu^\varepsilon \int \frac{\text{\dj}\omega\ \left[(\vec{k}-\vec{\omega})^2-\omega_4(\omega_4-k_4)\right]}{\vec{k}^2\omega^2(k-\omega)^2}\left[d-1-\frac{\vec{k}^2}{\vec{\omega}^2}+\frac{\vec{k}\cdot(\vec{k}-\vec{\omega})^2}{\vec{\omega}^2(\vec{k}-\vec{\omega})^2}\right], \tag{2.29}$$

$$\begin{aligned}t_{im}(\vec{k})\Gamma^{(1)}_{AA}(k_4^2,\vec{k}^2) + l_{im}(\vec{k})\overline{\Gamma}^{(1)}_{AA}(k_4^2,\vec{k}^2) &= N_c\mu^\varepsilon \int \frac{\text{\dj}\omega\ \vec{\omega}^2}{\vec{k}^2\omega^2(\vec{k}-\vec{\omega})^2}\left[\delta_{im}-\frac{\omega_i\omega_m}{\vec{\omega}^2}\right] \\ &\quad -\frac{1}{2}N_c\mu^\varepsilon \int \frac{\text{\dj}\omega\ t_{jl}(\vec{\omega})t_{nk}(\vec{k}-\vec{\omega})}{\vec{k}^2\omega^2(k-\omega)^2}\times \\ &\quad [(2\omega-k)_i\delta_{kj}-2k_k\delta_{ij}+2k_j\delta_{ik}]\,[(2\omega-k)_m\delta_{ln}+2k_l\delta_{mn}-2k_n\delta_{ml}]\,.\end{aligned} \tag{2.30}$$

In the equation, (2.22), for $\Gamma_{AA}$ (the gluon polarization), the first two integral terms (which contain the possible energy divergences at one-loop) cancel explicitly and this is due to the cancellation of the Faddeev-Popov determinant against the determinant arising from the Gauß' law constraint that is the raison d'être for the first order formalism used in this work [1].

Let us now discuss in more detail the equations (2.24–2.30). Under the transformation $\omega_4 \to -\omega_4$, the right-hand side of Eq. (2.24) and the second term of the right-hand side of Eq. (2.28) change sign which means that these integrals must vanish: i.e.,

$$\Gamma^{(1)}_{A\pi}(k_4^2,\vec{k}^2) = \overline{\Gamma}^{(1)}_{A\pi}(k_4^2,\vec{k}^2) = 0 \tag{2.31}$$

and

$$\Gamma^{(1)}_{A\sigma}(k_4^2,\vec{k}^2) = N_c\mu^\varepsilon \int \frac{\text{đ}\omega\ \omega_4\vec{k}\cdot(\vec{k}-2\vec{\omega})}{k_4\vec{k}^2\omega^2(k-\omega)^2}\left[d-1-\frac{\vec{k}^2}{\vec{\omega}^2}+\frac{\vec{k}\cdot(\vec{k}-\vec{\omega})^2}{\vec{\omega}^2(\vec{k}-\vec{\omega})^2}\right]. \tag{2.32}$$

Comparing equations (2.25) and (2.26), we have clearly that

$$\Gamma^{(1)}_{\pi\sigma}(k_4^2,\vec{k}^2) = \Gamma^{(1)}_{c}(\vec{k}^2) \tag{2.33}$$

which is also a consequence of the cancellation of the Faddeev-Popov determinant. The tensor equations, (2.27) and (2.30), must be decomposed to extract the dressing functions. We notice however that contracting Eq. (2.27) with $l_{ki}(\vec{k})$ gives us the more useful combination

$$\Gamma^{(1)}_{\pi\pi}(k_4^2,\vec{k}^2) + \overline{\Gamma}^{(1)}_{\pi\pi}(k_4^2,\vec{k}^2) = -\Gamma^{(1)}_{c}(\vec{k}^2). \tag{2.34}$$

On the other hand, contracting Eq. (2.27) with $t_{ki}(\vec{k})$ gives

$$(d-1)\Gamma^{(1)}_{\pi\pi}(k_4^2,\vec{k}^2) = N_c\mu^\varepsilon \int \frac{\text{đ}\omega}{\omega^2(\vec{k}-\vec{\omega})^2}\left[d-2+\frac{\vec{k}\cdot\vec{\omega}^2}{\vec{k}^2\vec{\omega}^2}\right]. \tag{2.35}$$

Turning to Eq. (2.30), contracting with $l_{mi}(\vec{k})$ gives us

$$\overline{\Gamma}^{(1)}_{AA}(k_4^2,\vec{k}^2) = N_c\mu^\varepsilon \int \frac{\text{đ}\omega\ \left(\vec{k}^2\vec{\omega}^2-\vec{k}\cdot\vec{\omega}^2\right)}{\vec{k}^4\omega^2(\vec{k}-\vec{\omega})^2} - \frac{N_c}{2}\mu^\varepsilon\int\frac{\text{đ}\omega\ \vec{k}\cdot(\vec{k}-2\vec{\omega})^2}{\vec{k}^4\omega^2(k-\omega)^2}\left[d-1-\frac{\vec{k}^2}{\vec{\omega}^2}+\frac{\vec{k}\cdot(\vec{k}-\vec{\omega})^2}{\vec{\omega}^2(\vec{k}-\vec{\omega})^2}\right], \tag{2.36}$$

whereas contraction with $t_{mi}(\vec{k})$ gives

$$\begin{aligned}(d-1)\Gamma^{(1)}_{AA}(k_4^2,\vec{k}^2) \ &= \ N_c\mu^\varepsilon\int\frac{\text{đ}\omega\ \vec{\omega}^2}{\vec{k}^2\omega^2(\vec{k}-\vec{\omega})^2}t_{mi}(\vec{k})t_{im}(\vec{\omega}) - \frac{N_c}{2}\mu^\varepsilon\int\frac{\text{đ}\omega}{\vec{k}^2\omega^2(k-\omega)^2}t_{mi}(\vec{k})t_{jl}(\vec{\omega})t_{nk}(\vec{k}-\vec{\omega})\times\\ &\quad [(2\omega-k)_i\delta_{kj}-2k_k\delta_{ij}+2k_j\delta_{ik}]\left[(2\omega-k)_m\delta_{ln}+2k_l\delta_{mn}-2k_n\delta_{ml}\right].\end{aligned} \tag{2.37}$$

Expanding the transverse projectors of Eq. (2.37), contracting indices and where possible canceling denominator factors, we get

$$\begin{aligned}(d-1)\Gamma^{(1)}_{AA}(k_4^2,\vec{k}^2) \ &= \ N_c\mu^\varepsilon\int\frac{\text{đ}\omega}{\vec{k}^2\omega^2(\vec{k}-\vec{\omega})^2}\left[(d-2)\vec{\omega}^2+\frac{\vec{k}\cdot\vec{\omega}^2}{\vec{k}^2}\right]\\ &\quad +N_c\mu^\varepsilon\int\frac{\text{đ}\omega}{\vec{k}^2\omega^2(k-\omega)^2}\left\{(6-4d)\vec{k}^2+2(d-1)\frac{\vec{k}\cdot\vec{\omega}^2}{\vec{k}^2}+2(1-d)\vec{\omega}^2\right.\\ &\quad \left.+\frac{1}{\vec{\omega}^2}\left[-\frac{1}{4}\vec{k}^4-\frac{1}{2}\vec{k}^2\vec{k}\cdot\vec{\omega}+(4d-5)\vec{k}\cdot\vec{\omega}^2-2\frac{\vec{k}\cdot\vec{\omega}^3}{\vec{k}^2}\right]+\frac{1}{8}\frac{\vec{k}^6}{\vec{\omega}^2(\vec{k}-\vec{\omega})^2}\right\}.\end{aligned} \tag{2.38}$$

In order to reduce the number of integrals that we need to compute, we use the identities

$$\begin{aligned}\vec{k}\cdot\vec{\omega} \ &= \ 1/2(k^2+\omega^2-(k-\omega)^2)-k_4\omega_4,\\ \omega_4^2 \ &= \ \omega^2-\vec{\omega}^2\end{aligned}$$

to get

$$\begin{aligned}(d-1)\Gamma^{(1)}_{AA}(k_4^2,\vec{k}^2) \ &= \ \frac{N_c}{8}\mu^\varepsilon\int\frac{\text{đ}\omega\ \vec{k}^4}{\omega^2(k-\omega)^2\vec{\omega}^2(\vec{k}-\vec{\omega})^2}+\left[1+(10-8d)\frac{k^2}{\vec{k}^2}+3\frac{k^4}{\vec{k}^4}\right]\frac{N_c}{2}\mu^\varepsilon\int\frac{\text{đ}\omega\ k_4\omega_4}{\omega^2(k-\omega)^2\vec{\omega}^2}\\ &\quad +\left[(4d-5)\frac{k^4}{\vec{k}^2}-k^2-\vec{k}^2-\frac{k^6}{\vec{k}^4}\right]\frac{N_c}{4}\mu^\varepsilon\int\frac{\text{đ}\omega}{\omega^2(k-\omega)^2\vec{\omega}^2}\\ &\quad +N_c\mu^\varepsilon\int\frac{\text{đ}\omega}{\omega^2(k-\omega)^2}\left[1+(3-4d)\frac{k^2}{\vec{k}^2}+2\frac{k^4}{\vec{k}^4}+2\frac{\vec{k}\cdot\vec{\omega}}{\vec{k}^2}\left(\frac{k^2}{\vec{k}^2}-1\right)+2(1-d)\left(\frac{\vec{\omega}^2}{\vec{k}^2}-\frac{\vec{k}\cdot\vec{\omega}^2}{\vec{k}^4}\right)\right]\\ &\quad +N_c\mu^\varepsilon\int\frac{\text{đ}\omega}{\omega^2(\vec{k}-\vec{\omega})^2}\left[-\frac{1}{4}+\left(3d-\frac{17}{4}\right)\frac{k^2}{\vec{k}^2}-\frac{5}{4}\frac{k^4}{\vec{k}^4}+\frac{\vec{k}\cdot\vec{\omega}}{\vec{k}^2}\left(\frac{7}{2}-2d+\frac{3}{2}\frac{k^2}{\vec{k}^2}\right)+(d-2)\frac{\vec{\omega}^2}{\vec{k}^2}\right].\end{aligned} \tag{2.39}$$

We must also deal with the integrals for the $\Gamma_{\sigma\sigma}$, $\Gamma_{A\sigma}$ and $\overline{\Gamma}_{AA}$ Green's functions, given by equations (2.29), (2.32) and (2.36). It turns out that all these expressions are related as we will now show. Starting with $\Gamma_{A\sigma}$, Eq. (2.32), we can easily show that

$$\Gamma^{(1)}_{A\sigma}(k_4^2,\vec{k}^2) = (d-1)N_c\mu^\varepsilon \int \frac{đ\omega\ \omega_4\vec{k}\cdot(\vec{k}-2\vec{\omega})}{k_4\vec{k}^2\omega^2(k-\omega)^2} + N_c\mu^\varepsilon \int \frac{đ\omega\ (2\omega_4-k_4)\vec{k}\cdot\vec{\omega}^2}{k_4\vec{k}^2\omega^2(k-\omega)^2\vec{\omega}^2}. \tag{2.40}$$

The second integral expression will be reduced below but it is convenient for now to leave it in its present form. Now consider Eq. (2.29). If we first recall that the last factor was originally written in symmetric form

$$t_{ij}(\vec{\omega})t_{ji}(\vec{k}-\vec{\omega}) = \left[d-1-\frac{\vec{k}^2}{\vec{\omega}^2}+\frac{\vec{k}\cdot(\vec{k}-\vec{\omega})^2}{\vec{\omega}^2(\vec{k}-\vec{\omega})^2}\right] \tag{2.41}$$

then it is easy to show that

$$\Gamma^{(1)}_{\sigma\sigma}(k_4^2,\vec{k}^2) = N_c\mu^\varepsilon \int \frac{đ\omega\ (k_4-2\omega_4)}{k_4\vec{k}^2\omega^2(k-\omega)^2}\left[(d-1)\vec{\omega}^2-\vec{k}^2\right] + N_c\mu^\varepsilon \int \frac{đ\omega\ (2\omega_4-k_4)\vec{k}\cdot\vec{\omega}^2}{k_4\vec{k}^2\omega^2(k-\omega)^2\vec{\omega}^2}. \tag{2.42}$$

We immediately recognize the latter factor as occurring in Eq. (2.40). As for the former, firstly we have as a general result that

$$\int \frac{đ\omega\ (k_4-2\omega_4)}{\omega^2(k-\omega)^2} = 0. \tag{2.43}$$

It can also be shown (using the results of Appendix A) that

$$\int \frac{đ\omega\ (k_4-2\omega_4)\vec{\omega}^2}{k_4\vec{k}^2\omega^2(k-\omega)^2} = \int \frac{đ\omega\ \omega_4\vec{k}\cdot(\vec{k}-2\vec{\omega})}{k_4\vec{k}^2\omega^2(k-\omega)^2}. \tag{2.44}$$

Thus we see that

$$\Gamma^{(1)}_{\sigma\sigma}(k_4^2,\vec{k}^2) = \Gamma^{(1)}_{A\sigma}(k_4^2,\vec{k}^2). \tag{2.45}$$

Turning now to Eq. (2.36), it is easy to show that

$$\overline{\Gamma}^{(1)}_{AA}(k_4^2,\vec{k}^2) = -N_c\mu^\varepsilon\frac{(d-1)}{2}\int \frac{đ\omega\ \vec{k}\cdot(\vec{k}-2\vec{\omega})^2}{\vec{k}^4\omega^2(k-\omega)^2} - N_c\mu^\varepsilon\int \frac{đ\omega\ k_4(2\omega_4-k_4)\vec{k}\cdot\vec{\omega}^2}{\vec{k}^4\omega^2(k-\omega)^2\vec{\omega}^2} \tag{2.46}$$

and further manipulating, we get that

$$\overline{\Gamma}^{(1)}_{AA}(k_4^2,\vec{k}^2) = -\frac{k_4^2}{\vec{k}^2}\Gamma^{(1)}_{A\sigma}(k_4^2,\vec{k}^2). \tag{2.47}$$

Let us finally reduce the expression for $\Gamma_{A\sigma}$, Eq. (2.40), to the set of most basic integrals as we did for $\Gamma_{AA}$. Using the same techniques as before, we have that

$$\begin{aligned}\Gamma^{(1)}_{A\sigma}(k_4^2,\vec{k}^2) \;=\; & \left[\frac{k^2}{\vec{k}^2}+\frac{1}{2}\frac{k^4}{k_4^2\vec{k}^2}\right]N_c\mu^\varepsilon\int\frac{đ\omega\ k_4\omega_4}{\omega^2(k-\omega)^2\vec{\omega}^2} - \frac{1}{4}\frac{k^4}{\vec{k}^2}N_c\mu^\varepsilon\int\frac{đ\omega}{\omega^2(k-\omega)^2\vec{\omega}^2}\\ & +N_c\mu^\varepsilon\int\frac{đ\omega}{\omega^2(k-\omega)^2}\left[(d-1)\frac{\omega_4}{k_4}-1+2\frac{k^2}{\vec{k}^2}-2(d-1)\frac{\omega_4\vec{k}\cdot\vec{\omega}}{k_4\vec{k}^2}+2\frac{\vec{k}\cdot\vec{\omega}}{\vec{k}^2}\right]\\ & +N_c\mu^\varepsilon\int\frac{đ\omega}{\omega^2(\vec{k}-\vec{\omega})^2}\left[-1-\frac{5}{4}\frac{k^2}{\vec{k}^2}+\frac{3}{2}\frac{\vec{k}\cdot\vec{\omega}}{\vec{k}^2}\right].\end{aligned} \tag{2.48}$$

Before proceeding, let us briefly summarize the results of this section. We have written the various one-loop, two-point proper Green's functions in terms of the most simple collection of integrals. $\Gamma^{(1)}_{A\pi}$ and $\overline{\Gamma}^{(1)}_{A\pi}$ are trivial, Eq. (2.31). We must calculate $\Gamma^{(1)}_{c}$, Eq. (2.25), which will also yield $\Gamma^{(1)}_{\pi\sigma}$ via Eq. (2.33). With $\Gamma^{(1)}_{\pi\pi}$ calculated using Eq. (2.35), we also get $\overline{\Gamma}^{(1)}_{\pi\pi}$ from Eq. (2.34). $\Gamma^{(1)}_{A\sigma}$ is calculated using Eq. (2.48) which then gives us $\Gamma^{(1)}_{\sigma\sigma}$ and $\overline{\Gamma}^{(1)}_{AA}$ from relations (2.45) and (2.47), respectively. Finally, we have to calculate $\Gamma^{(1)}_{AA}$ using Eq. (2.39). The necessary integrals will be derived in the next section.

## 3. NON-COVARIANT LOOP INTEGRALS

In this section we consider the nontrivial loop integrals that arise in this study (i.e., massless one-loop two-point integrals). Quite generally, the integrals that arise within Coulomb gauge perturbation theory can be classified into two categories – those that can be evaluated using standard techniques such as Feynman parametrization or Schwinger parameters (and which are detailed in Appendix A) and those that cannot. The latter category clearly requires a different approach and to this effect we derive a technique based on differential equations and integration by parts [IBP] suitable for the non-covariant setting here. We consider the three integrals:

$$A(k_4^2,\vec{k}^2) = \int \frac{đ\omega}{\omega^2(k-\omega)^2\vec{\omega}^2}, \tag{3.1}$$
$$A^4(k_4^2,\vec{k}^2) = \int \frac{đ\omega\ \omega_4}{\omega^2(k-\omega)^2\vec{\omega}^2}, \tag{3.2}$$
$$B(k_4^2,\vec{k}^2) = \int \frac{đ\omega}{\omega^2(k-\omega)^2\vec{\omega}^2(\vec{k}-\vec{\omega})^2}. \tag{3.3}$$

It is convenient to introduce the following notation: $x = k_4^2$, $y = \vec{k}^2$, $z = x/y$, $v = y/x$. We find that the above integrals can be written in the form

$$A(x,y) = \frac{(x+y)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} f_a(z), \tag{3.4}$$
$$A^4(x,y) = k_4\overline{A}(x,y) = k_4\frac{(x+y)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} f_4(z), \tag{3.5}$$
$$B(x,y) = \frac{y^{-2}(x+y)^{-\varepsilon}}{(4\pi)^{2-\varepsilon}} f_b(z) \tag{3.6}$$

where the $f_i(z)$ are functions of $z$ and which may contain an ultraviolet divergence in the form of a simple pole as $\varepsilon \to 0$. The functions $f_i(z)$ can (and will) be written down in analytic form for $\varepsilon \to 0$, but it turns out to be more useful to write some of the various parts in terms of an integral representation and also as asymptotic series in $z$ or $v$ which are more amenable to eventual numerical evaluation.

### A. Derivation of the Differential Equations

To begin, let us derive the differential equations that these integrals obey. Consider the general integral ($n = 0, 1$)

$$I^n(k_4^2,\vec{k}^2) = \int \frac{đ\omega\ \omega_4^n}{\omega^2(k-\omega)^2\vec{\omega}^2}. \tag{3.7}$$

Since $I^n$ is a function of two variables, there are two first derivatives:

$$k_4\frac{\partial I^n}{\partial k_4} = \int \frac{đ\omega\ \omega_4^n}{\omega^2(k-\omega)^2\vec{\omega}^2}\left\{-2\frac{k_4(k_4-\omega_4)}{(k-\omega)^2}\right\}, \tag{3.8}$$
$$k_k\frac{\partial I^n}{\partial k_k} = \int \frac{đ\omega\ \omega_4^n}{\omega^2(k-\omega)^2\vec{\omega}^2}\left\{-2\frac{\vec{k}\cdot(\vec{k}-\vec{\omega})}{(k-\omega)^2}\right\}. \tag{3.9}$$

Now, there are also two integration by parts identities:

$$0 = \int đ\omega\frac{\partial}{\partial\omega_4}\frac{\omega_4^{n+1}}{\omega^2(k-\omega)^2\vec{\omega}^2} = \int \frac{đ\omega\ \omega_4^n}{\omega^2(k-\omega)^2\vec{\omega}^2}\left\{n+1-2\frac{\omega_4^2}{\omega^2}-2\frac{\omega_4(\omega_4-k_4)}{(k-\omega)^2}\right\}, \tag{3.10}$$
$$0 = \int đ\omega\frac{\partial}{\partial\omega_i}\frac{\omega_i\omega_4^n}{\omega^2(k-\omega)^2\vec{\omega}^2} = \int \frac{đ\omega\ \omega_4^n}{\omega^2(k-\omega)^2\vec{\omega}^2}\left\{d-2-2\frac{\vec{\omega}^2}{\omega^2}-2\frac{\vec{\omega}\cdot(\vec{\omega}-\vec{k})}{(k-\omega)^2}\right\}. \tag{3.11}$$

Adding these two expressions gives

$$0 = \int \frac{đ\omega\ \omega_4^n}{\omega^2(k-\omega)^2\vec{\omega}^2}\left\{d+n-5+2\frac{k\cdot(k-\omega)}{(k-\omega)^2}\right\} \tag{3.12}$$

from which we have the important identity

$$k_4 \frac{\partial I^n}{\partial k_4} + k_k \frac{\partial I^n}{\partial k_k} = (d+n-5) I^n. \tag{3.13}$$

Expanding the numerator factor, Eq. (3.12) can be rewritten

$$0 = \int \frac{d\!\!\!{}^{-}\omega \; \omega_4^n}{\omega^2 (k-\omega)^2 \vec{\omega}^2} \left\{ d+n-4+\frac{k^2-\omega^2}{(k-\omega)^2} \right\}. \tag{3.14}$$

Similarly, Eq. (3.10) becomes

$$0 = \int \frac{d\!\!\!{}^{-}\omega \; \omega_4^n}{\omega^2 (k-\omega)^2 \vec{\omega}^2} \left\{ n-1+2\frac{\vec{\omega}^2}{\omega^2} + \frac{\left[-(k_4-\omega_4)^2 + k_4^2 - \omega^2 + \vec{\omega}^2\right]}{(k-\omega)^2} \right\}. \tag{3.15}$$

We can now rewrite Eq. (3.8) as

$$k_4 \frac{\partial I^n}{\partial k_4} = \int \frac{d\!\!\!{}^{-}\omega \; \omega_4^n}{\omega^2 (k-\omega)^2 \vec{\omega}^2} \left\{ 1-n+\frac{k_4^2}{k^2}\left[d+n-4\right] - 2\frac{\vec{\omega}^2}{\omega^2} + 2\frac{\vec{k}^2}{k^2}\frac{\omega^2}{(k-\omega)^2} - 2\frac{\vec{\omega}^2}{(k-\omega)^2} \right\} \tag{3.16}$$

and so we arrive at the temporal differential equations for $A$ and $A^4$:

$$\begin{aligned} k_4 \frac{\partial A}{\partial k_4} &= \left[1+2(d-4)\frac{k_4^2}{k^2}\right] A + 2\frac{\vec{k}^2}{k^2} \int \frac{d\!\!\!{}^{-}\omega}{(k-\omega)^4 \vec{\omega}^2} - 2\int \frac{d\!\!\!{}^{-}\omega}{\omega^4 (k-\omega)^2} - 2\int \frac{d\!\!\!{}^{-}\omega}{\omega^2 (k-\omega)^4}, && (3.17) \\ k_4 \frac{\partial A^4}{\partial k_4} &= \left[2(d-3)\frac{k_4^2}{k^2}\right] A^4 + 2\frac{\vec{k}^2}{k^2} \int \frac{d\!\!\!{}^{-}\omega \; \omega_4}{(k-\omega)^4 \vec{\omega}^2} - 2\int \frac{d\!\!\!{}^{-}\omega \; \omega_4}{\omega^4 (k-\omega)^2} - 2\int \frac{d\!\!\!{}^{-}\omega \; \omega_4}{\omega^2 (k-\omega)^4}. && (3.18) \end{aligned}$$

The differential equations involving the spatial components are subsequently given using Eq. (3.13).

Now let us consider integrals of the form $B$. Again, we have two IBP identities:

$$\begin{aligned} 0 = \int d\!\!\!{}^{-}\omega \frac{\partial}{\partial \omega_4} \frac{\omega_4}{\omega^2 (k-\omega)^2 \vec{\omega}^2 (\vec{k}-2\vec{\omega})^2} &= \int \frac{d\!\!\!{}^{-}\omega}{\omega^2 (k-\omega)^2 \vec{\omega}^2 (\vec{k}-2\vec{\omega})^2} \left\{ 1-2\frac{\omega_4^2}{\omega^2} - 2\frac{\omega_4(\omega_4-k_4)}{(k-\omega)^2} \right\}, && (3.19) \\ 0 = \int d\!\!\!{}^{-}\omega \frac{\partial}{\partial \omega_i} \frac{\omega_i}{\omega^2 (k-\omega)^2 \vec{\omega}^2 (\vec{k}-2\vec{\omega})^2} &= \int \frac{d\!\!\!{}^{-}\omega}{\omega^2 (k-\omega)^2 \vec{\omega}^2 (\vec{k}-2\vec{\omega})^2} \left\{ d-2-2\frac{\vec{\omega}^2}{\omega^2} - 2\frac{\vec{\omega}\cdot(\vec{\omega}-\vec{k})}{(k-\omega)^2} - 2\frac{\vec{\omega}\cdot(\vec{\omega}-\vec{k})}{(\vec{k}-\vec{\omega})^2} \right\}, && (3.20) \end{aligned}$$

and by adding the two we see that

$$0 = \int \frac{d\!\!\!{}^{-}\omega}{\omega^2 (k-\omega)^2 \vec{\omega}^2 (\vec{k}-2\vec{\omega})^2} \left\{ d-7+2\frac{k\cdot(k-\omega)}{(k-\omega)^2} + 2\frac{\vec{k}\cdot(\vec{k}-\vec{\omega})}{(\vec{k}-\vec{\omega})^2} \right\} \tag{3.21}$$

from which we have the important identity

$$k_4 \frac{\partial B}{\partial k_4} + k_k \frac{\partial B}{\partial k_k} = (d-7) B. \tag{3.22}$$

Rewriting the definition of $B$ as

$$2B = 2\int \frac{d\!\!\!{}^{-}\omega \; w_4^2}{\omega^4 (k-\omega)^2 \vec{\omega}^2 (\vec{k}-2\vec{\omega})^2} + 2\int \frac{d\!\!\!{}^{-}\omega}{\omega^4 (k-\omega)^2 (\vec{k}-\vec{\omega})^2} \tag{3.23}$$

and using Eq. (3.19) gives

$$\begin{aligned} B - 2\int \frac{d\!\!\!{}^{-}\omega}{\omega^2 (k-\omega)^4 \vec{\omega}^2} &= \int \frac{d\!\!\!{}^{-}\omega}{\omega^2 (k-\omega)^2 \vec{\omega}^2 (\vec{k}-2\vec{\omega})^2} \left\{ -2\frac{\omega_4(\omega_4-k_4)}{(k-\omega)^2} \right\} \\ &= \int \frac{d\!\!\!{}^{-}\omega}{\omega^2 (k-\omega)^2 \vec{\omega}^2 (\vec{k}-2\vec{\omega})^2} \left\{ -2\frac{(\omega_4-k_4)^2}{(k-\omega)^2} + 2\frac{k_4(k_4-\omega)}{(k-\omega)^2} \right\} \\ &= -2B + 2\int \frac{d\!\!\!{}^{-}\omega}{\omega^2 (k-\omega)^4 \vec{\omega}^2} - k_4 \frac{\partial B}{\partial k_4}. && (3.24) \end{aligned}$$

However, with Eq. (3.14) we obtain the temporal differential equation for $B$:

$$k_4 \frac{\partial B}{\partial k_4} = -3B + \frac{4(4-d)}{k^2} A + \frac{4}{k^2} \int \frac{\bar{d}\omega}{\omega^4 (\vec{k} - \vec{\omega})^4} \tag{3.25}$$

and the spatial equation is given via Eq. (3.22).

The differential equations are best written by evaluating the standard integrals in terms of $\varepsilon$ (see Appendix A) and with the notation $x$ and $y$ for the temporal and spatial momentum components. The result is ($A^4 = k_4 \overline{A}$):

$$\begin{aligned}
2x \frac{\partial A}{\partial x} &= \left[1 - (2+4\varepsilon)\frac{x}{x+y}\right] A - 2\frac{y^{-\varepsilon}}{x+y} X + 4(x+y)^{-1-\varepsilon} Y, && (3.26)\\
2x \frac{\partial \overline{A}}{\partial x} &= \left[-1 - 4\varepsilon\frac{x}{x+y}\right] \overline{A} - 2\frac{y^{-\varepsilon}}{x+y} X + 2(x+y)^{-1-\varepsilon} Y, && (3.27)\\
2x \frac{\partial B}{\partial x} &= -3B + (4+8\varepsilon)(x+y)^{-1} A - 4\frac{y^{-1-\varepsilon}}{x+y} X, && (3.28)
\end{aligned}$$

where

$$X = \frac{1}{(4\pi)^{2-\varepsilon}} \Gamma(\varepsilon)\Gamma(1-\varepsilon) \frac{\Gamma(1/2-\varepsilon)}{\Gamma(1/2-2\varepsilon)}, \quad Y = \frac{1}{(4\pi)^{2-\varepsilon}} \Gamma(\varepsilon)\Gamma(1-\varepsilon) \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}. \tag{3.29}$$

The spatial differential equations subsequently read:

$$\begin{aligned}
2y \frac{\partial A}{\partial y} &= \left[-3 - 2\varepsilon + (2+4\varepsilon)\frac{x}{x+y}\right] A + 2\frac{y^{-\varepsilon}}{x+y} X - 4(x+y)^{-1-\varepsilon} Y, && (3.30)\\
2y \frac{\partial \overline{A}}{\partial y} &= \left[-1 - 2\varepsilon + 4\varepsilon\frac{x}{x+y}\right] \overline{A} + 2\frac{y^{-\varepsilon}}{x+y} X - 2(x+y)^{-1-\varepsilon} Y, && (3.31)\\
2y \frac{\partial B}{\partial y} &= (-1 - 2\varepsilon) B - (4+8\varepsilon)(x+y)^{-1} A + 4\frac{y^{-1-\varepsilon}}{x+y} X. && (3.32)
\end{aligned}$$

## B. Solving the Differential Equations

The differential equations (3.26-3.28, 3.30-3.32) have a rather special structure that allows for a relatively simple solution. Taking $A$ to start (the other two follow the same pattern), we can write

$$A(x,y) = F_A(x,y) G_A(x,y) \tag{3.33}$$

such that

$$\begin{aligned}
2x \frac{\partial F_A(x,y)}{\partial x} &= \left[1 - (2+4\varepsilon)\frac{x}{x+y}\right] F_A(x,y), && (3.34)\\
2y \frac{\partial F_A(x,y)}{\partial y} &= \left[-3 - 2\varepsilon + (2+4\varepsilon)\frac{x}{x+y}\right] F_A(x,y), && (3.35)\\
F_A(x,y) 2x \frac{\partial G_A(x,y)}{\partial x} &= -2\frac{y^{-\varepsilon}}{x+y} X + 4(x+y)^{-1-\varepsilon} Y, && (3.36)\\
F_A(x,y) 2y \frac{\partial G_A(x,y)}{\partial y} &= 2\frac{y^{-\varepsilon}}{x+y} X - 4(x+y)^{-1-\varepsilon} Y. && (3.37)
\end{aligned}$$

The solution for the first pair of equations can be written down without difficulty and is

$$F_A(x,y) = x^{1/2} y^{-1/2+\varepsilon} (x+y)^{-1-2\varepsilon} + \mathcal{C}. \tag{3.38}$$

By inspection, the constant $\mathcal{C} = 0$ since the function $F_A(x,y)$ has the dimension $[y]^{-\varepsilon}$ (i.e., the function $F_A$ carries the dimension of the integral $A$). The remaining differential equations for $G_A$ now read

$$\begin{aligned}
x^{1/2} y^{-1/2+\varepsilon} (x+y)^{-1-2\varepsilon} 2x \frac{\partial G_A(x,y)}{\partial x} &= -2\frac{y^{-\varepsilon}}{x+y} X + 4(x+y)^{-1-\varepsilon} Y, && (3.39)\\
x^{1/2} y^{-1/2+\varepsilon} (x+y)^{-1-2\varepsilon} 2y \frac{\partial G_A(x,y)}{\partial y} &= 2\frac{y^{-\varepsilon}}{x+y} X - 4(x+y)^{-1-\varepsilon} Y. && (3.40)
\end{aligned}$$

Because the derivatives with respect to $x$ and $y$ are distinguished only by a minus sign, the function $G_A$ must be a dimensionless function of the dimensionless ratio $v = y/x$ or (equivalently) $z = x/y$. We choose to express $G_A$ as a function of $v$ to avoid singularities and the two partial differential equations collapse into a single first order ordinary differential equation:

$$\frac{dG_A(v)}{dv} = v^{-1/2-2\varepsilon}(1+v)^{2\varepsilon}X - 2v^{-1/2-\varepsilon}(1+v)^{\varepsilon}Y. \tag{3.41}$$

Using the integral representation of the hypergeometric function [16]:

$$\int_0^v dt\, t^{b-1}(1+t)^{-a} = \frac{1}{b}\frac{v^b}{(1+v)^a}\,{}_2F_1\left(a,1;1+b;\frac{v}{1+v}\right), \tag{3.42}$$

gives the solution

$$\begin{aligned} G_A(v) - G_A(0) \;=\; & \frac{1}{1/2-2\varepsilon}\frac{v^{1/2-2\varepsilon}}{(1+v)^{-2\varepsilon}}X\,{}_2F_1\left(-2\varepsilon,1;3/2-2\varepsilon;\frac{v}{1+v}\right) \\ & -\frac{2}{1/2-\varepsilon}\frac{v^{1/2-\varepsilon}}{(1+v)^{-\varepsilon}}X\,{}_2F_1\left(-\varepsilon,1;3/2-\varepsilon;\frac{v}{1+v}\right). \end{aligned} \tag{3.43}$$

Let us now show that the constant $G_A(0) = 0$. The general solution for $A(x,y)$ as $y \to 0$ would have a term

$$F_A(x, y \to 0)G_A(0) \sim \lim_{y\to 0} x^{1/2}y^{-1/2+\varepsilon} \tag{3.44}$$

but in the original partial differential equation (3.26), which must still be defined for all values of $y$ (although the coefficients may be singular) there is no such term and the constant $G_A(0)$ must vanish. We thus have the solution

$$\begin{aligned} A(x,y) = & \frac{1}{(x+y)^{1+\varepsilon}}\times \\ & \left\{\frac{X}{1/2-2\varepsilon}\left(\frac{v}{1+v}\right)^{-e}{}_2F_1\left(-2\varepsilon,1;3/2-2\varepsilon;\frac{v}{1+v}\right) - \frac{2Y}{1/2-\varepsilon}{}_2F_1\left(-\varepsilon,1;3/2-\varepsilon;\frac{v}{1+v}\right)\right\}. \end{aligned} \tag{3.45}$$

As it stands, the solution is not of much use since the hypergeometric functions are somewhat cumbersome. However, we are primarily interested in the solution as $\varepsilon \to 0$. In this case

$${}_2F_1\left(-\varepsilon,1;3/2-\varepsilon;\frac{v}{1+v}\right) = 1 + \varepsilon f(v) + \mathcal{O}(\varepsilon^2) \tag{3.46}$$

and upon expanding $A$ in powers of $\varepsilon$ the functions $f(v)$ happen to cancel such that

$$A(x,y) = \frac{(x+y)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{-2\left(\frac{1}{\varepsilon}-\gamma\right) - 4\ln 2 + 2\ln\left(\frac{x+y}{y}\right) + \mathcal{O}(\varepsilon)\right\}. \tag{3.47}$$

A few remarks are in order here. The overall dimension of the integral $A$ can be written in terms of the covariant factor $x+y$ as for the integrals in linear covariant gauges and result in the standard logarithmic factor, singular on the light-cone and with a branch-cut extending into the timelike region. The ultraviolet divergence characterized by the $1/\varepsilon$ term is similarly covariant. The noncovariant component is logarithmically singular at $y = 0$. The $x$-dependence of the noncovariant component, however, has the logarithmic singularity at $x + y = 0$. This is actually rather crucial since otherwise it would be difficult to justify the analytic continuation between Euclidean and Minkowski space without further cancellations in forming the two-point functions. Lastly, it is remarkable that the integral $A$ which defies standard evaluation techniques reduces merely to a combination of logarithms. Unfortunately, this simplicity will not be present in the integrals $A^4$ and $B$.

Let us now turn to the function $\overline{A}$ governed by equations (3.27) and (3.31). Writing $\overline{A}(x,y) = F_{\overline{A}}G_{\overline{A}}$ as before leads to four partial differential equations:

$$\begin{aligned} 2x\frac{\partial F_{\overline{A}}(x,y)}{\partial x} &= \left[1 - 4\varepsilon\frac{x}{x+y}\right]F_{\overline{A}}(x,y), & (3.48)\\ 2y\frac{\partial F_{\overline{A}}(x,y)}{\partial y} &= \left[-1 - 2\varepsilon + 4\varepsilon\frac{x}{x+y}\right]F_{\overline{A}}(x,y), & (3.49)\\ F_{\overline{A}}(x,y)2x\frac{\partial G_{\overline{A}}(x,y)}{\partial x} &= -2\frac{y^{-\varepsilon}}{x+y}X + 2(x+y)^{-1-\varepsilon}Y, & (3.50)\\ F_{\overline{A}}(x,y)2y\frac{\partial G_{\overline{A}}(x,y)}{\partial y} &= 2\frac{y^{-\varepsilon}}{x+y}X - 2(x+y)^{-1-\varepsilon}Y. & (3.51) \end{aligned}$$

The solution to the first pair is

$$F_{\overline{A}}(x,y) = x^{-1/2}y^{-1/2+\varepsilon}(x+y)^{-2\varepsilon}, \tag{3.52}$$

the constant vanishing on dimensional grounds as previously. The latter two differential equations now read

$$\begin{aligned} x^{-1/2}y^{-1/2+\varepsilon}(x+y)^{-2\varepsilon}2x\frac{\partial G_{\overline{A}}(x,y)}{\partial x} &= -2\frac{y^{-\varepsilon}}{x+y}X + 2(x+y)^{-1-\varepsilon}Y, &(3.53)\\ x^{-1/2}y^{-1/2+\varepsilon}(x+y)^{-2\varepsilon}2y\frac{\partial G_{\overline{A}}(x,y)}{\partial y} &= 2\frac{y^{-\varepsilon}}{x+y}X - 2(x+y)^{-1-\varepsilon}Y. &(3.54)\end{aligned}$$

Again, as before the relative minus sign means that $G_{\overline{A}}$ is a function of the ratio, but this time it is useful to use both $v$ and $z$, the reason becoming clear shortly. In terms of $v$, we have that

$$\frac{dG_{\overline{A}}(v)}{dv} = v^{-1/2-2\varepsilon}(1+v)^{-1+2\varepsilon}X - v^{-1/2-\varepsilon}(1+v)^{-1+\varepsilon}Y \tag{3.55}$$

whose solution is

$$G_{\overline{A}}(v) = \frac{X}{1/2-2\varepsilon}\frac{v^{1/2-2\varepsilon}}{(1+v)^{1-2\varepsilon}}{}_2F_1\left(1-2\varepsilon,1;3/2-2\varepsilon;\frac{v}{1+v}\right) - \frac{Y}{1/2-\varepsilon}\frac{v^{1/2-\varepsilon}}{(1+v)^{1-\varepsilon}}{}_2F_1\left(1-\varepsilon,1;3/2-\varepsilon;\frac{v}{1+v}\right), \tag{3.56}$$

whereas in terms of $z$ we get that

$$\frac{dG_{\overline{A}}(z)}{dz} = -z^{-1/2}(1+z)^{-1+2\varepsilon}X + z^{-1/2}(1+z)^{-1+\varepsilon}Y \tag{3.57}$$

whose solution is

$$G_{\overline{A}}(z) = -\frac{2X}{1/2-2\varepsilon}\frac{z^{1/2}}{(1+z)^{1-2\varepsilon}}{}_2F_1\left(1-2\varepsilon,1;3/2;\frac{z}{1+z}\right) + \frac{2Y}{1/2-\varepsilon}\frac{z^{1/2}}{(1+z)^{1-\varepsilon}}{}_2F_1\left(1-\varepsilon,1;3/2;\frac{z}{1+z}\right). \tag{3.58}$$

The general constant terms in both solutions vanish as before. We can now write the result for the full integral $A^4(x,y) = k_4\overline{A}$ in two ways:

$$\begin{aligned} A^4(x,y) &= \frac{k_4}{(x+y)^{1+\varepsilon}}\times\\ &\left\{\frac{X}{1/2-2\varepsilon}\left(\frac{v}{1+v}\right)^{-e}{}_2F_1\left(1-2\varepsilon,1;3/2-2\varepsilon;\frac{v}{1+v}\right) - \frac{Y}{1/2-\varepsilon}{}_2F_1\left(1-\varepsilon,1;3/2-\varepsilon;\frac{v}{1+v}\right)\right\}, &(3.59)\\ A^4(x,y) &= \frac{k_4}{(x+y)^{1+\varepsilon}}\left\{-2X(1+z)^{\varepsilon}{}_2F_1\left(1-2\varepsilon,1;3/2;\frac{z}{1+z}\right) + 2Y{}_2F_1\left(1-\varepsilon,1;3/2;\frac{z}{1+z}\right)\right\}. &(3.60)\end{aligned}$$

Now, in contradistinction to the integral $A$, the expansion of the hypergeometric functions in powers of $\varepsilon$ does not yield a simple result – quite the contrary. However, one can expand in powers of either $v$ or $z$ and then take the limit $\varepsilon \to 0$ which will prove useful in later numerical evaluation. Actually, in the case of the expansion with $v$, one must first collect together terms involving $\ln v/(1+v)$ since these factors do not have an expansion around $v = 0$. The results are ultraviolet finite and are:

$$\begin{aligned} A^4(x,y) &\overset{v\to 0}{=} k_4\frac{(x+y)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{\left(\ln\left(\frac{v}{1+v}\right)+2\ln 2\right)\left[-2-\frac{4}{3}v+\frac{4}{15}v^2-\frac{4}{35}v^3+\frac{4}{63}v^4+\ldots\right]\right.\\ &\left.\qquad +4+\frac{20}{9}v-\frac{124}{225}v^2+\frac{988}{3675}v^3-\frac{3244}{19845}v^4+\ldots\right\}, &(3.61)\\ A^4(x,y) &\overset{z\to 0}{=} k_4\frac{(x+y)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{\ln 2\left[4+\frac{8}{3}z-\frac{8}{15}z^2+\frac{8}{35}z^3-\frac{8}{63}z^4+\ldots\right]-\frac{2}{3}z-\frac{1}{15}z^2+\frac{8}{105}z^3-\frac{23}{378}z^4+\ldots\right\}. &(3.62)\end{aligned}$$

The statement that the expansion of the hypergeometric functions in powers of $\varepsilon$ is nontrivial is fortunately not the whole story. It is possible to find the full solution as $\varepsilon \to 0$ by rewriting the hypergeometric functions in their integral

representation, expanding the integrand and exploring whether or not it is possible to evaluate the resulting integrals. In the case of using $v$ as the variable this is not the case – the expression reads:

$$A^4(x,y) = k_4 \frac{(x+y)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{(1+v)}{\sqrt{v}} \left\{ -4\ln 2 \arctan\left(\sqrt{v}\right) - \int_0^v \frac{dt}{\sqrt{t}(1+t)} \ln\left(\frac{t}{1+t}\right) \right\} \tag{3.63}$$

and the integral cannot be done directly in terms of known functions (or to phrase it more properly, the direct result is not known to the authors at present although indirectly the result can be inferred from the following discussion). However, with $z$ as the variable, we get

$$A^4(x,y) = k_4 \frac{(x+y)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}} \frac{(1+z)}{\sqrt{z}} \left\{ 4\ln 2 \arctan\left(\sqrt{z}\right) - \int_0^z \frac{dt}{\sqrt{t}(1+t)} \ln(1+t) \right\} \tag{3.64}$$

and the integral is

$$\begin{aligned} \int_0^z \frac{dt}{\sqrt{t}(1+t)} \ln(1+t) &= \pi \ln 2 - \imath \ln\left(\sqrt{z} - \imath\right) \left[ \ln 2 + \ln(1+z) - \ln\left(1 - \imath\sqrt{z}\right) - \frac{1}{2}\ln\left(\sqrt{z} - \imath\right) \right] \\ &\quad + \imath \ln\left(\sqrt{z} + \imath\right) \left[ \ln 2 + \ln(1+z) - \ln\left(1 + \imath\sqrt{z}\right) - \frac{1}{2}\ln\left(\sqrt{z} + \imath\right) \right] \\ &\quad - \imath \mathrm{dilog}\left(\frac{1}{2} - \frac{\imath}{2}\sqrt{z}\right) + \imath \mathrm{dilog}\left(\frac{1}{2} + \frac{\imath}{2}\sqrt{z}\right), \end{aligned} \tag{3.65}$$

in terms of the dilogarithmic function [16]. An expansion in powers of $z$ yields the expansion Eq. (3.62) which serves as a useful check. Whilst it is gratifying that the integral can eventually be written in terms of known functions, these functions must still be evaluated at some stage. For this reason, the integral form Eq. (3.64) and the asymptotic forms Eqs. (3.61,3.62) are actually of more practical use.

It is possible to discuss the analytic structure of $A^4$. Clearly, it is ultraviolet finite. With the expansion Eq. (3.61) in $v = y/x$ we see that as $y \to 0$ there is a logarithmic singularity as seen previously for the integral $A$. As for the singularities in $x$, by rewriting the integral

$$\frac{1}{\sqrt{z}} \int_0^z \frac{dt}{\sqrt{t}(1+t)} \ln(1+t) = \int_0^1 \frac{dt}{\sqrt{t}(1+zt)} \ln(1+zt) \tag{3.66}$$

and knowing the analytic properties of $\arctan\left(\sqrt{z}\right)/\sqrt{z}$ we see that the singularity occurs for $z = -1$ (i.e., for $x+y = 0$) with branch cuts extending into the timelike region. Again, this behavior is the same as for $A$.

Finally let us discuss the integral $B(x,y)$ satisfying equations (3.28) and (3.32). We separate the function into two parts as previously:

$$B(x,y) = F_B(x,y) G_B(x,y) \tag{3.67}$$

such that $F_B$ and $G_B$ obey the following differential equations:

$$2x \frac{\partial F_B(x,y)}{\partial x} = -3F_B(x,y), \tag{3.68}$$

$$2y \frac{\partial F_B(x,y)}{\partial y} = -(1+2\varepsilon) F_B(x,y), \tag{3.69}$$

$$F_B(x,y) 2x \frac{\partial G_B(x,y)}{\partial x} = (4+8\varepsilon)(x+y)^{-1} A(x,y) - 4\frac{y^{-1-\varepsilon}}{x+y} X, \tag{3.70}$$

$$F_B(x,y) 2y \frac{\partial G_B(x,y)}{\partial y} = -(4+8\varepsilon)(x+y)^{-1} A(x,y) + 4\frac{y^{-1-\varepsilon}}{x+y} X. \tag{3.71}$$

The solution to the first pair of differential equations is

$$F_B(x,y) = x^{-3/2} y^{-1/2-\varepsilon} \tag{3.72}$$

with the possible constant vanishing as before on dimensional grounds. The second pair of equations now reads

$$x^{-3/2} y^{-1/2-\varepsilon} 2x \frac{\partial G_B(x,y)}{\partial x} = (4+8\varepsilon)(x+y)^{-1} A(x,y) - 4\frac{y^{-1-\varepsilon}}{x+y} X, \tag{3.73}$$

$$x^{-3/2} y^{-1/2-\varepsilon} 2y \frac{\partial G_B(x,y)}{\partial y} = -(4+8\varepsilon)(x+y)^{-1} A(x,y) + 4\frac{y^{-1-\varepsilon}}{x+y} X. \tag{3.74}$$

We are now faced with a potential problem – for general $\varepsilon$, the function $A(x,y)$ is itself a combination of hypergeometric functions and we have little chance of solving the resulting differential equations. However, we can write down $A(x,y)$ for vanishing $\varepsilon$. Again, the derivatives with respect to $x$ and $y$ are distinguished by a minus sign and we can rewrite the two equations as a first order differential equation in $z = x/y$. In this case, we cannot use the variable $v$ since this leads to non-integrable singularities. After expanding in powers of $\varepsilon$, the equation is

$$\begin{aligned}\frac{dG_B(z)}{dz} &= \frac{2(1+2\varepsilon)}{(4\pi)^{2-\varepsilon}} z^{1/2}(1+z)^{-2}\left[-2\left(\frac{1}{\varepsilon}-\gamma\right) - 4\ln 2 + 4\ln(1+z) + \mathcal{O}(\varepsilon)\right] \\ &\quad - \frac{2}{(4\pi)^{2-\varepsilon}} z^{1/2}(1+z)^{-1}\left[\frac{1}{\varepsilon} - \gamma - 2\ln 2 + \mathcal{O}(\varepsilon)\right]. \end{aligned} \tag{3.75}$$

Since there is no interference between the integration over $z$ and the expansion in $\varepsilon$, we do not need to consider the terms of $\mathcal{O}(\varepsilon)$ further. We thus obtain

$$\begin{aligned} &G_B(z) \\ &= \frac{1}{(4\pi)^{2-\varepsilon}}\left\{-\frac{4z^{3/2}}{(1+z)}\left(\frac{1}{\varepsilon}-\gamma-2\ln 2\right) + 8(1+2\ln 2)\left[\frac{\sqrt{z}}{(1+z)} - \arctan(\sqrt{z})\right] + 8\int_0^z \frac{dt\,\sqrt{t}}{(1+t)^2}\ln(1+t) + \mathcal{O}(\varepsilon)\right\} \end{aligned} \tag{3.76}$$

where the constant $G_B(0)$ vanishes as before. Using integration by parts, we have that

$$\int_0^z \frac{dt\,\sqrt{t}}{(1+t)^2}\ln(1+t) = \frac{1}{2}\int_0^z \frac{dt}{\sqrt{t}(1+t)}\ln(1+t) - \frac{\sqrt{z}}{(1+z)}\ln(1+z) - \frac{\sqrt{z}}{(1+z)} + \arctan(\sqrt{z}) \tag{3.77}$$

and so, the full solution can be written

$$\begin{aligned} B(x,y) &= \frac{(x+y)^{-\varepsilon}y^{-2}}{(4\pi)^{2-\varepsilon}}\left\{-4(1+z)^{-1}\left(\frac{1}{\varepsilon}-\gamma-2\ln 2+\ln(1+z)\right) + 16\ln 2\left[\frac{1}{z(1+z)} - \frac{1}{z^{3/2}}\arctan(\sqrt{z})\right]\right. \\ &\quad \left. - \frac{8}{z(1+z)}\ln(1+z) + \frac{4}{z}\int_0^1 \frac{dt}{\sqrt{t}(1+zt)}\ln(1+zt) + \mathcal{O}(\varepsilon)\right\}. \end{aligned} \tag{3.78}$$

The factor $y^{-2}$ is extracted globally for convenience as it happens that the integral $B(x,y)$ is multiplied by $y^2$ when forming the gluon polarization. Without the factor $y^{-2}$, the expression is finite as $y \to 0$. The singularities in $x$ occur for $z = -1$ as before, with branch cuts extending into the timelike region. An expansion around $z = 0$ is possible and reads

$$\begin{aligned} B(x,y) &\overset{z\to 0}{=} \frac{(x+y)^{-\varepsilon}y^{-2}}{(4\pi)^{2-\varepsilon}}\left\{-4\left(\frac{1}{\varepsilon}-\gamma\right)\left[1-z+z^2-z^3+z^4+\ldots\right] + 8\ln 2\left[-\frac{1}{3}+\frac{3}{5}z-\frac{5}{7}z^2+\frac{7}{9}z^3-\frac{9}{11}z^4+\ldots\right]\right. \\ &\quad \left. + \left[-\frac{16}{3}+\frac{28}{5}z-\frac{46}{7}z^2+\frac{202}{27}z^3-\frac{91}{11}z^4+\ldots\right]\right\}. \end{aligned} \tag{3.79}$$

Comparing Eq. (3.78) with Eq. (3.64) and using the known expansion for $A_4(x,y)$ for small $v$, Eq. (3.61), we can also expand $B(x,y)$ and the result reads:

$$\begin{aligned} B(x,y) &\overset{v\to 0}{=} \frac{(x+y)^{-\varepsilon}y^{-2}}{(4\pi)^{2-\varepsilon}}\left\{-4\left(\frac{1}{\varepsilon}-\gamma\right)\left[v-v^2+v^3-v^4+\ldots\right]\right. \\ &\quad \left. +4\left(\ln\left(\frac{v}{1+v}\right)+2\ln 2\right)\left[v+v^2-\frac{5}{3}v^3+\frac{7}{5}v^4+\ldots\right] - 16v^2 + \frac{64}{9}v^3 - \frac{368}{75}v^4+\ldots\right\}. \end{aligned} \tag{3.80}$$

Having derived the integrals, it is pertinent to see if they can be checked using standard techniques. Since this is a somewhat technical exercise that does not add to the discussion here, we present the details in Appendix B. It is also useful to plot the integrals and their asymptotic expansions. The functions $f_i(z)$, defined in Eqs. (3.4-3.6), and with the factors proportional to $(1/\varepsilon - \gamma)$ removed are presented in Figures 1-3. All functions are monotonically increasing with $z$. One can see that the asymptotic expansions do indeed represent the functions within their domains of applicability. The functions $f_a(z)$ and $f_4(z)$ are logarithmically singular as $y \to 0$ ($z \to \infty$ with $x$ fixed), whereas $f_b(z)$ is finite. All functions exhibit singularities as $z \to -1$ (especially strong in the case of $f_b$ although this can be attributed to the different choice of prefactor in the full integral $B$). However most importantly, there are no singularities at $z = 0$ (or equivalently $x = 0$ for finite $y$), such that the continuation to spacelike Minkowski space is entirely justified.

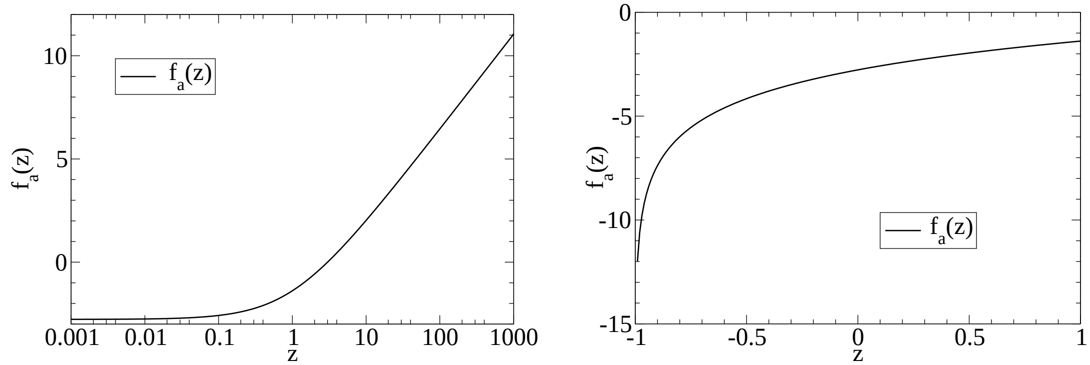


FIG. 1: UV-finite part of the function $f_a(z)$. Left panel: Euclidean values of $z$. Right panel: continuation of $f_a(z)$ into the spacelike Minkowski region.

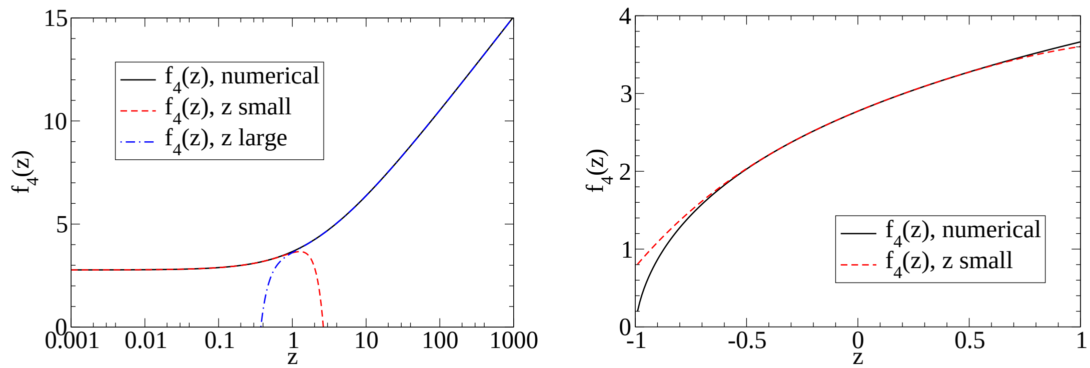


FIG. 2: The function $f_4(z)$ and its asymptotic expansions for $z \to 0$ and $v = 1/z \to 0$. Left panel: Euclidean values of $z$. Right panel: continuation of $f_4(z)$ into the spacelike Minkowski region.

## 4. ONE-LOOP TWO-POINT FUNCTIONS

Having now evaluated all the integrals that occur (see the previous section and Appendix A), we can now return to the expressions for the one-loop, proper two-point functions and write out our results. To do this, it is convenient to define two combinations of functions:

$$\begin{aligned} f(z) &= 4\ln 2 \frac{1}{\sqrt{z}} \arctan\sqrt{z} - \int_0^1 \frac{dt}{\sqrt{t}(1+zt)} \ln(1+zt), \\ g(z) &= 2\ln 2 - \ln(1+z). \end{aligned} \tag{4.1}$$

[These functions are actually variations of the finite parts of $f_4$ and $f_a$ from before.] The results are:

$$\Gamma^{(1)}_{A\pi}(x,y) = \overline{\Gamma}^{(1)}_{A\pi}(x,y) = 0, \tag{4.2}$$

$$\Gamma^{(1)}_{\pi\sigma}(x,y) = \Gamma^{(1)}_{c}(y) = \frac{N_c}{(4\pi)^{2-\varepsilon}} \left\{ -\frac{4}{3}\left[\frac{1}{\varepsilon} - \gamma - \ln\left(\frac{y}{\mu}\right)\right] - \frac{28}{9} + \frac{8}{3}\ln 2 + \mathcal{O}(\varepsilon) \right\}, \tag{4.3}$$

$$\Gamma^{(1)}_{\pi\pi}(x,y) = \frac{N_c}{(4\pi)^{2-\varepsilon}} \left\{ \frac{4}{3}\left[\frac{1}{\varepsilon} - \gamma - \ln\left(\frac{y}{\mu}\right)\right] + \frac{52}{9} - \frac{8}{3}\ln 2 + \mathcal{O}(\varepsilon) \right\}, \tag{4.4}$$

$$\overline{\Gamma}^{(1)}_{\pi\pi}(x,y) = \frac{N_c}{(4\pi)^{2-\varepsilon}} \left\{ -\frac{8}{3} + \mathcal{O}(\varepsilon) \right\}, \tag{4.5}$$

$$\begin{aligned} \Gamma^{(1)}_{A\sigma}(x,y) &= \Gamma^{(1)}_{\sigma\sigma}(x,y) = -\frac{y}{x}\overline{\Gamma}_{AA}(x,y) \\ &= \frac{N_c}{(4\pi)^{2-\varepsilon}} \left\{ -\frac{1}{3}\left[\frac{1}{\varepsilon} - \gamma - \ln\left(\frac{x+y}{\mu}\right)\right] + \frac{1}{9} - 6(1+z) + 3(1+z)g(z) + \frac{1}{2}(1+z)(1+3z)f(z) + \mathcal{O}(\varepsilon) \right\}, \end{aligned}$$

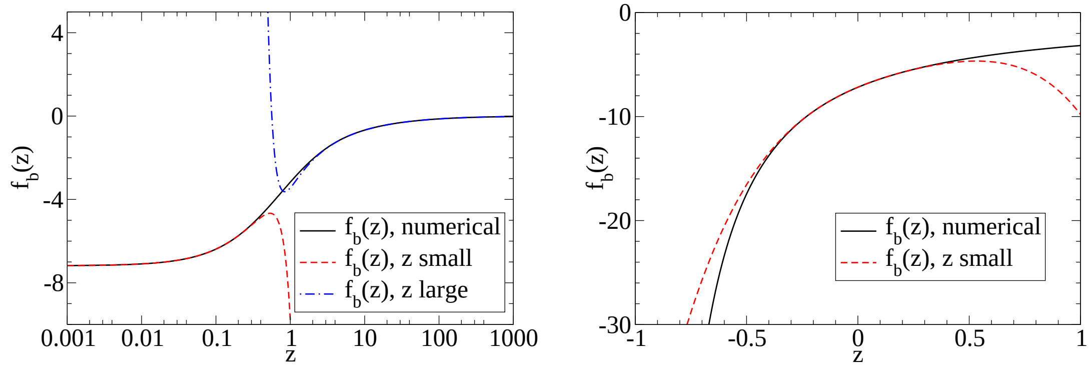


FIG. 3: UV-finite part of the function $f_b(z)$ and its asymptotic expansions for $z \to 0$ and $v = 1/z \to 0$. Left panel: Euclidean values of $z$. Right panel: continuation of $f_b(z)$ into the spacelike Minkowski region.

$$ \tag{4.6} $$

$$\begin{aligned} \Gamma_{AA}^{(1)}(x,y) &= \frac{N_c}{(4\pi)^{2-\varepsilon}} \left\{ \left(-1+\frac{1}{3}z\right)\left[\frac{1}{\varepsilon}-\gamma-\ln\left(\frac{x+y}{\mu}\right)\right] - \frac{52}{9} + \frac{116}{9}(1+z) - 3z(1+z) + \frac{4}{3}g(z) \right. \\ & \left. +(1+z)\left(\frac{1}{2z}-6+\frac{3}{2}z\right)g(z) + (1+z)\left(-\frac{1}{4z}+\frac{1}{4}-\frac{11}{4}z+\frac{3}{4}z^2\right)f(z) + \mathcal{O}(\varepsilon)\right\}. \end{aligned} \tag{4.7}$$

It is immediately apparent that for finite $y$, there are no singularities in any of the above at $x = 0$ (or equivalently, $z = 0$) since $f(0) = 2g(0) = 4\ln 2$ cancels the $1/z$ pole. This is to be expected since none of the individual integrals or their prefactors are singular at this point.

Let us now construct the one-loop propagator dressing functions. Writing $D_{\alpha\beta} = D^{(0)}_{\alpha\beta} + g^2 D^{(1)}_{\alpha\beta}$ and using Eqs. (2.31,2.33,2.45), Eq. (2.11) becomes

$$\begin{aligned} D_{AA}^{(1)}(x,y) &= \Gamma_{\pi\pi}^{(1)}(x,y) - \frac{1}{(1+z)}\left[\Gamma_{AA}^{(1)}(x,y)+\Gamma_{\pi\pi}^{(1)}(x,y)\right], \\ D_{\pi\pi}^{(1)}(x,y) &= \Gamma_{AA}^{(1)}(x,y) - \frac{1}{(1+z)}\left[\Gamma_{AA}^{(1)}(x,y)+\Gamma_{\pi\pi}^{(1)}(x,y)\right], \\ D_{A\pi}^{(1)}(x,y) &= -\frac{1}{(1+z)}\left[\Gamma_{AA}^{(1)}(x,y)+\Gamma_{\pi\pi}^{(1)}(x,y)\right], \\ D_{\sigma\sigma}^{(1)}(x,y) &= \Gamma_{A\sigma}^{(1)}(x,y) - 3\Gamma_c^{(1)}(y), \\ D_{\phi\phi}^{(1)}(x,y) &= -\Gamma_{A\sigma}^{(1)}(x,y), \\ D_{\sigma\phi}^{(1)}(x,y) &= \Gamma_{A\sigma}^{(1)}(x,y) - \Gamma_c^{(1)}(y), \\ D_c^{(1)}(y) &= -\Gamma_c^{(1)}(y), \\ D_{\sigma\lambda}^{(1)}(x,y) &= -\Gamma_c^{(1)}(y), \\ D_{\phi\lambda}^{(1)}(x,y) &= 0. \end{aligned} \tag{4.8}$$

The last equation of (2.11) becomes an identity by virtue of the relation Eq. (2.47), as it should. Putting in the above results for the one-loop, two-point proper functions, we have (in Euclidean space and in the limit $\varepsilon \to 0$):

$$D_{AA}^{(1)}(x,y) = \frac{N_c}{(4\pi)^{2-\varepsilon}}\left\{\left[\frac{1}{\varepsilon}-\gamma-\ln\left(\frac{x+y}{\mu}\right)\right] - \frac{64}{9} + 3z + \left[-\frac{1}{2z}+\frac{14}{3}-\frac{3}{2}z\right]g(z) + \left[\frac{1}{4z}-\frac{1}{4}+\frac{11}{4}z-\frac{3}{4}z^2\right]f(z)\right\}, \tag{4.9}$$

$$\begin{aligned} D_{\pi\pi}^{(1)}(x,y) = \frac{N_c}{(4\pi)^{2-\varepsilon}} &\left\{\left(-\frac{4}{3}+\frac{1}{3}z\right)\left[\frac{1}{\varepsilon}-\gamma-\ln\left(\frac{x+y}{\mu}\right)\right] - \frac{52}{9} + \frac{116}{9} - 3z^2 + \left[\frac{11}{6}-6z+\frac{3}{2}z^2\right]g(z)\right. \\ & \left. + \left[-\frac{1}{4}-\frac{1}{4}z-\frac{11}{4}z^2+\frac{3}{4}z^3\right]f(z)\right\}, \end{aligned} \tag{4.10}$$

$$D^{(1)}_{A\pi}(x,y)=\frac{N_c}{(4\pi)^{2-\varepsilon}}\left\{-\frac{1}{3}\left[\frac{1}{\varepsilon}-\gamma-\ln\left(\frac{x+y}{\mu}\right)\right]-\frac{116}{9}+3z+\left[-\frac{1}{2z}+6-\frac{3}{2}z\right]g(z)\right.$$
$$\left.+\left[\frac{1}{4z}-\frac{1}{4}+\frac{11}{4}z-\frac{3}{4}z^2\right]f(z)\right\}, \tag{4.11}$$

$$D^{(1)}_{\sigma\sigma}(x,y)=\frac{N_c}{(4\pi)^{2-\varepsilon}}\left\{\frac{11}{3}\left[\frac{1}{\varepsilon}-\gamma-\ln\left(\frac{x+y}{\mu}\right)\right]+\frac{31}{9}-6z+(-1+3z)g(z)+\frac{1}{2}(1+z)(1+3z)f(z)\right\}, \tag{4.12}$$

$$D^{(1)}_{\phi\phi}(x,y)=\frac{N_c}{(4\pi)^{2-\varepsilon}}\left\{\frac{1}{3}\left[\frac{1}{\varepsilon}-\gamma-\ln\left(\frac{x+y}{\mu}\right)\right]+\frac{53}{9}+6z-3(1+z)g(z)-\frac{1}{2}(1+z)(1+3z)f(z)\right\}, \tag{4.13}$$

$$D^{(1)}_{\sigma\phi}(x,y)=\frac{N_c}{(4\pi)^{2-\varepsilon}}\left\{\left[\frac{1}{\varepsilon}-\gamma-\ln\left(\frac{x+y}{\mu}\right)\right]-\frac{25}{9}-6z+\left[\frac{5}{3}+3z\right]g(z)+\frac{1}{2}(1+z)(1+3z)f(z)\right\}, \tag{4.14}$$

$$D^{(1)}_{c}(y)=D^{(1)}_{\sigma\lambda}(x,y)=\frac{N_c}{(4\pi)^{2-\varepsilon}}\left\{\frac{4}{3}\left[\frac{1}{\varepsilon}-\gamma-\ln\left(\frac{y}{\mu}\right)\right]+\frac{28}{9}-\frac{8}{3}\ln 2\right\}, \tag{4.15}$$

$$D^{(1)}_{\phi\lambda}(y)=0. \tag{4.16}$$

A few remarks are in order here. The momentum dependence of the relationship between the vector propagators and proper two-point functions, Eq. (2.11), is such that the only occurrence of a momentum dependent UV-divergence is within the $D_{\pi\pi}$ propagator – the factor $1/(1+z)$ is otherwise canceled within the combination $\Gamma^{(1)}_{AA}+\Gamma^{(1)}_{\pi\pi}$. Indeed, the divergence structure of the one-loop, proper two-point functions has been known for some time [1]. The momentum dependent coefficient of the $1/\varepsilon$-pole in $D^{(1)}_{\pi\pi}$ is symptomatic of the fact that the $\pi$-field is not multiplicatively renormalizable since the $\pi$-field has its origins in the linearization of the (composite) chromoelectric field term of the action that is central to the first order formalism. However, we are further able to see that for the UV-finite parts, the kinematical singularities on the light-cone ($z=-1$) reside purely in the logarithmic term and the functions $f(z)$ and $g(z)$ which are logarithmic in character. There are no singularities in the Euclidean or spacelike Minkowski regions ($z>-1$). Hence, we can conclude that the analytic continuations between Euclidean and Minkowski space have entirely the same character as in linear covariant gauges – that is to say that the continuation is justified.

Although it is not our intention to discuss the renormalization aspects of the two-point Green's functions in Coulomb gauge, at the one-loop perturbative level it is possible to identify two renormalization group invariant combinations of propagator dressing functions via the coefficients of the $1/\varepsilon$ poles. In Landau gauge, a renormalization group invariant running coupling may be defined through the combination of gluon and ghost propagator dressing functions: $g^2D_{AA}D_c^2$, [17]. This stems from the Slavnov–Taylor identity which expresses the universality of the coupling and the Landau gauge property that the ghost-gluon vertex is UV-finite. Given that at one-loop, the coefficient of the $1/\varepsilon$ pole of $g^2$ is the first coefficient of the $\beta$-function ($b_0=-11N_c/3(4\pi)^{2-\varepsilon}$) and is gauge invariant, the combination $D_{AA}D_c^2$ in Landau gauge has a $1/\varepsilon$ pole with the coefficient $11N_c/3(4\pi)^{2-\varepsilon}$. In Coulomb gauge, the results above clearly show the same result. However, the individual coefficients for $D_{AA}$ and $D_c$ are different from Landau gauge. The second renormalization group invariant combination of propagator dressing functions is particular to Coulomb gauge and is $g^2D_{\sigma\sigma}$, [1]. The coefficient of the $1/\varepsilon$ pole in $D_{\sigma\sigma}$ above clearly confirms this.

## 5. SUMMARY AND OUTLOOK

A one-loop perturbative analysis of Coulomb gauge Yang-Mills theory within the first order formalism has been undertaken. The various propagator and two-point proper dressing functions have been explicitly evaluated at this order. In order to do this, dimensionally regularized results for the noncovariant two-point loop integrals inherent to Coulomb gauge have been derived using techniques based on differential equations and integration by parts identities.

The results for the two-point functions are rather interesting. The dressing functions are dimensionless functions of two independent variables and a mass scale introduced via the regularization. These functions can be split into two parts – one unambiguously connected to the UV-divergence involving also the logarithmic behavior normally associated with covariant gauges and a second part which is a UV-finite function of the ratio of the temporal to spatial components of the momentum.

The analytic continuations between Minkowski and Euclidean space within the noncovariant setting can be justified on the grounds that the possible singularities occur on the light-cone, with branch cuts extending into the timelike Minkowski region. It is seen explicitly that for spacelike Minkowski and Euclidean momenta there are no singularities in any of the two-point Green's functions, which is as it should be.

The outlook for future work done in Coulomb gauge is rather promising. The most direct continuation of this work is to consider the vertex functions of the theory. A generalization of the differential equation technique to the various

one-loop three-point loop integrals certainly appears feasible albeit challenging. Subsequently, a two-loop perturbative analysis studying, for example, the cancellation of potentially energy divergent integrals or the renormalization would be of great interest. A second area of interest would be to include quarks and to study physical high energy processes, as has been done in linear covariant gauges. The relationship between the covariant and noncovariant descriptions of the same phenomena will undoubtedly lead to greater insight into the physical mechanisms at work.

One of the motivations for studying Coulomb gauge is that nonperturbative phenomena such as confinement and bound states may be better understood in this gauge. The analysis of nonperturbative physics is however greatly constrained by the perturbative behavior. As an example, consider the evaluation of nonperturbative loop integrals – the phase space of the integration measure still contains the perturbative domain and renormalization is still necessary despite the fact that one may be considering infrared external momentum scales. Further, the techniques developed here to evaluate the perturbative integrals will almost certainly be of help when studying the Dyson–Schwinger equations nonperturbatively. Also, the perturbative expansion (although asymptotic) is of great use in verifying nonperturbative identities such as the Slavnov–Taylor identities. The Slavnov–Taylor identities are the focus of present work [18].

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## APPENDIX A: STANDARD INTEGRALS

There are two particular types of integral that we wish to consider in this appendix and both can be done using standard techniques. We use the Schwinger parametrization method [11]. The integrals considered have two or three denominator factors (with arbitrary powers), at least one of which contains both spatial and temporal components. We list results for all the possible vector and tensor integrals that arise.

Consider then the integral (in Euclidean space)

$$I = \int \frac{\bar{d}\omega}{[\omega^2]^{\mu}\,[(k-\omega)^2]^{\nu}}. \tag{A.1}$$

Using the identity [16]

$$\frac{1}{a^{\nu}} = \frac{1}{\Gamma(\nu)}\int_0^{\infty} d\alpha\,\alpha^{\nu-1}\exp\{-\alpha a\}, \quad \Re\nu > 0, \;\; \Re a > 0 \tag{A.2}$$

we have

$$I = \frac{1}{\Gamma(\mu)\Gamma(\nu)}\int_0^{\infty} d\alpha d\beta\,\alpha^{\mu-1}\beta^{\nu-1}\int \bar{d}\omega\;\exp\left\{-(\alpha+\beta)w^2 + 2\beta k\cdot\omega - \beta k^2\right\}. \tag{A.3}$$

Shifting variables

$$\omega \to \omega + \frac{\beta}{\alpha+\beta}k \tag{A.4}$$

gives

$$I = \frac{1}{\Gamma(\mu)\Gamma(\nu)}\int_0^{\infty} d\alpha d\beta\,\alpha^{\mu-1}\beta^{\nu-1}\int \bar{d}\omega\;\exp\left\{-(\alpha+\beta)w^2 - \frac{\alpha\beta}{\alpha+\beta}k^2\right\} \tag{A.5}$$

and rescaling, $\omega \to (\alpha+\beta)^{-1/2}\omega$, leads us to

$$I = \frac{1}{\Gamma(\mu)\Gamma(\nu)}\int_0^{\infty} d\alpha d\beta\,\alpha^{\mu-1}\beta^{\nu-1}(\alpha+\beta)^{\varepsilon-2}\exp\left\{-\frac{\alpha\beta}{\alpha+\beta}k^2\right\}\int \bar{d}\omega\;\exp\left\{-w^2\right\}. \tag{A.6}$$

The integral over $\omega$ can now be carried out:

$$I = \frac{1}{(4\pi)^{2-\varepsilon}}\frac{1}{\Gamma(\mu)\Gamma(\nu)}\int_0^{\infty} d\alpha d\beta\,\alpha^{\mu-1}\beta^{\nu-1}(\alpha+\beta)^{\varepsilon-2}\exp\left\{-\frac{\alpha\beta}{\alpha+\beta}k^2\right\}. \tag{A.7}$$

By inserting the identity $1=\int_0^\infty d\lambda\,\delta(\lambda-\alpha-\beta)$ and rescaling $\alpha\to\lambda\alpha$, $\beta\to\lambda\beta$ we have

$$\begin{aligned} I &= \frac{1}{(4\pi)^{2-\varepsilon}}\frac{1}{\Gamma(\mu)\Gamma(\nu)}\int_0^1 d\alpha d\beta\,\delta(1-\alpha-\beta)\alpha^{\mu-1}\beta^{\nu-1}(\alpha+\beta)^{\varepsilon-2}\int_0^\infty d\lambda\,\lambda^{\mu+\nu+\varepsilon-3}\exp\left\{-\lambda\frac{\alpha\beta}{\alpha+\beta}k^2\right\}, \\ &= \frac{1}{(4\pi)^{2-\varepsilon}}\frac{1}{\Gamma(\mu)\Gamma(\nu)}\int_0^1 d\alpha\,\alpha^{\mu-1}(1-\alpha)^{\nu-1}\int_0^\infty d\lambda\,\lambda^{\mu+\nu+\varepsilon-3}\exp\left\{-\lambda\alpha(1-\alpha)k^2\right\}. \end{aligned} \tag{A.8}$$

The integral over $\lambda$ can be done and is a variation of Eq. (A.2) giving

$$I=\frac{1}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(\mu+\nu+\varepsilon-2)}{\Gamma(\mu)\Gamma(\nu)}\int_0^1 d\alpha\,\alpha^{\mu-1}(1-\alpha)^{\nu-1}\left[\alpha(1-\alpha)k^2\right]^{2-\mu-\nu-\varepsilon}. \tag{A.9}$$

Finally the integral over $\alpha$ can be done (it has the integral form of the beta-function) to give the final result:

$$I=\int\frac{\text{đ}\omega}{\left[\omega^2\right]^\mu\left[(k-\omega)^2\right]^\nu}=\frac{\left[k^2\right]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(\mu+\nu+\varepsilon-2)}{\Gamma(\mu)\Gamma(\nu)}\frac{\Gamma(2-\mu-\varepsilon)\Gamma(2-\nu-\varepsilon)}{\Gamma(4-\mu-\nu-2\varepsilon)}. \tag{A.10}$$

Now let us consider the vector integral

$$I=\int\frac{\text{đ}\omega\ \omega_4}{\left[\omega^2\right]^\mu\left[(k-\omega)^2\right]^\nu}. \tag{A.11}$$

In order to proceed we notice the following. Under the change of variables $\omega\to\omega+k\beta/(\alpha+\beta)$ we have:

$$\int\text{đ}\omega\ \exp\left\{-(\alpha+\beta)w^2+2\beta k\cdot\omega-\beta k^2\right\}=\int\text{đ}\omega\ \exp\left\{-(\alpha+\beta)w^2-\frac{\alpha\beta}{\alpha+\beta}k^2\right\}. \tag{A.12}$$

Differentiating with respect to $k_4$ gives

$$2\beta\int\text{đ}\omega\,(\omega_4-k_4)\exp\left\{-(\alpha+\beta)w^2+2\beta k\cdot\omega-\beta k^2\right\}=-2\beta\int\text{đ}\omega\,\frac{\alpha}{\alpha+\beta}k_4\exp\left\{-(\alpha+\beta)w^2-\frac{\alpha\beta}{\alpha+\beta}k^2\right\} \tag{A.13}$$

which shows us that

$$\int\text{đ}\omega\ \omega_4\exp\left\{-(\alpha+\beta)w^2+2\beta k\cdot\omega-\beta k^2\right\}=\int\text{đ}\omega\,\frac{\beta}{\alpha+\beta}k_4\exp\left\{-(\alpha+\beta)w^2-\frac{\alpha\beta}{\alpha+\beta}k^2\right\}. \tag{A.14}$$

Further differentiation gives rise to expressions for integrals involving other numerator structures. Proceeding as before, we have the results

$$\int\frac{\text{đ}\omega\ \omega_4}{\left[\omega^2\right]^\mu\left[(k-\omega)^2\right]^\nu} = k_4\frac{\left[k^2\right]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(\mu+\nu+\varepsilon-2)}{\Gamma(\mu)\Gamma(\nu)}\frac{\Gamma(3-\mu-\varepsilon)\Gamma(2-\nu-\varepsilon)}{\Gamma(5-\mu-\nu-2\varepsilon)}, \tag{A.15}$$

$$\int\frac{\text{đ}\omega\ \omega_i}{\left[\omega^2\right]^\mu\left[(k-\omega)^2\right]^\nu} = k_i\frac{\left[k^2\right]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(\mu+\nu+\varepsilon-2)}{\Gamma(\mu)\Gamma(\nu)}\frac{\Gamma(3-\mu-\varepsilon)\Gamma(2-\nu-\varepsilon)}{\Gamma(5-\mu-\nu-2\varepsilon)}, \tag{A.16}$$

$$\begin{aligned}\int\frac{\text{đ}\omega\ \omega_4^2}{\left[\omega^2\right]^\mu\left[(k-\omega)^2\right]^\nu} &= \frac{\left[k^2\right]^{3-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(\mu+\nu+\varepsilon-3)}{\Gamma(\mu)\Gamma(\nu)}\frac{\Gamma(3-\mu-\varepsilon)\Gamma(2-\nu-\varepsilon)}{\Gamma(6-\mu-\nu-2\varepsilon)}\times \\ &\quad\left\{\frac{1}{2}(2-\nu-\varepsilon)+\frac{k_4^2}{k^2}(\mu+\nu+\varepsilon-3)(3-\mu-\varepsilon)\right\},\end{aligned} \tag{A.17}$$

$$\begin{aligned}\int\frac{\text{đ}\omega\ \omega_i\omega_j}{\left[\omega^2\right]^\mu\left[(k-\omega)^2\right]^\nu} &= \frac{\left[k^2\right]^{3-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(\mu+\nu+\varepsilon-3)}{\Gamma(\mu)\Gamma(\nu)}\frac{\Gamma(3-\mu-\varepsilon)\Gamma(2-\nu-\varepsilon)}{\Gamma(6-\mu-\nu-2\varepsilon)}\times \\ &\quad\left\{\frac{1}{2}\delta_{ij}(2-\nu-\varepsilon)+\frac{k_ik_j}{k^2}(\mu+\nu+\varepsilon-3)(3-\mu-\varepsilon)\right\},\end{aligned} \tag{A.18}$$

$$\int\frac{\text{đ}\omega\ \omega_4\omega_i}{\left[\omega^2\right]^\mu\left[(k-\omega)^2\right]^\nu} = k_4k_i\frac{\left[k^2\right]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(\mu+\nu+\varepsilon-2)}{\Gamma(\mu)\Gamma(\nu)}\frac{\Gamma(4-\mu-\varepsilon)\Gamma(2-\nu-\varepsilon)}{\Gamma(6-\mu-\nu-2\varepsilon)}, \tag{A.19}$$

$$\begin{aligned}\int\frac{\text{đ}\omega\ \omega_4\omega_i\omega_j}{\left[\omega^2\right]^\mu\left[(k-\omega)^2\right]^\nu} &= k_4\frac{\left[k^2\right]^{3-\mu-\nu-\varepsilon}}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(\mu+\nu+\varepsilon-3)}{\Gamma(\mu)\Gamma(\nu)}\frac{\Gamma(4-\mu-\varepsilon)\Gamma(2-\nu-\varepsilon)}{\Gamma(7-\mu-\nu-2\varepsilon)}\times \\ &\quad\left\{\frac{1}{2}\delta_{ij}(2-\nu-\varepsilon)+\frac{k_ik_j}{k^2}(\mu+\nu+\varepsilon-3)(4-\mu-\varepsilon)\right\}.\end{aligned} \tag{A.20}$$

Using exactly the same techniques we have

$$\int \frac{d\!\!\bar{}\,\omega}{[\omega^2]^\mu \left[(\vec{k}-\vec{w})^2\right]^\nu} = \frac{\left[\vec{k}^2\right]^{2-\mu-\nu-\varepsilon}}{(4\pi)^{2-e}} \frac{\Gamma(\mu+\nu+\varepsilon-2)}{\Gamma(\mu)\Gamma(\nu)} \frac{\Gamma(3/2-\nu-\varepsilon)\Gamma(2-\mu-\varepsilon)}{\Gamma(7/2-\mu-\nu-2\varepsilon)}, \tag{A.21}$$

$$\int \frac{d\!\!\bar{}\,\omega}{[\omega^2]^\mu \left[(\vec{k}-\vec{w})^2\right]^\nu [\vec{\omega}^2]^\rho} = \frac{\left[\vec{k}^2\right]^{2-\mu-\nu-\rho-\varepsilon}}{(4\pi)^{2-e}} \frac{\Gamma(\mu+\nu+\rho+\varepsilon-2)}{\Gamma(\mu)\Gamma(\nu)} \frac{\Gamma(\mu-1/2)}{\Gamma(\mu+\rho-1/2)} \frac{\Gamma(2-\mu-\rho-\varepsilon)\Gamma(3/2-\nu-\varepsilon)}{\Gamma(7/2-\mu-\nu-\rho-2\varepsilon)}, \tag{A.22}$$

$$\int \frac{d\!\!\bar{}\,\omega\ \omega_4^n}{[\omega^2]^\mu \left[(\vec{k}-\vec{w})^2\right]^\nu [\vec{\omega}^2]^\rho} = 0 \quad (n,\ \text{odd}), \tag{A.23}$$

$$\int \frac{d\!\!\bar{}\,\omega\ \omega_i}{[\omega^2]^\mu \left[(\vec{k}-\vec{w})^2\right]^\nu [\vec{\omega}^2]^\rho} = k_i \frac{\left[\vec{k}^2\right]^{2-\mu-\nu-\rho-\varepsilon}}{(4\pi)^{2-e}} \frac{\Gamma(\mu+\nu+\rho+\varepsilon-2)}{\Gamma(\mu)\Gamma(\nu)} \frac{\Gamma(\mu-1/2)}{\Gamma(\mu+\rho-1/2)} \frac{\Gamma(3-\mu-\rho-\varepsilon)\Gamma(3/2-\nu-\varepsilon)}{\Gamma(9/2-\mu-\nu-\rho-2\varepsilon)}, \tag{A.24}$$

$$\begin{aligned}\int \frac{d\!\!\bar{}\,\omega\ \omega_i\omega_j}{[\omega^2]^\mu \left[(\vec{k}-\vec{w})^2\right]^\nu [\vec{\omega}^2]^\rho} &= \frac{\left[\vec{k}^2\right]^{3-\mu-\nu-\rho-\varepsilon}}{(4\pi)^{2-e}} \frac{\Gamma(\mu+\nu+\rho+\varepsilon-3)}{\Gamma(\mu)\Gamma(\nu)} \frac{\Gamma(\mu-1/2)}{\Gamma(\mu+\rho-1/2)} \frac{\Gamma(3-\mu-\rho-\varepsilon)\Gamma(3/2-\nu-\varepsilon)}{\Gamma(11/2-\mu-\nu-\rho-2\varepsilon)} \times \\ &\quad \left\{\delta_{ij}\frac{1}{2}(3/2-\nu-\varepsilon) + \frac{k_ik_j}{\vec{k}^2}(\mu+\nu+\rho-3+\varepsilon)(3-\mu-\rho-\varepsilon)\right\}.\end{aligned} \tag{A.25}$$

## APPENDIX B: CHECKING THE NONSTANDARD INTEGRALS

Since the integrals $A$, $A^4$ and $B$ must be derived using nonstandard techniques, it is worthwhile checking them where possible against available results. It turns out that the expansions around $z=0$, Eqs. (3.47,3.62,3.79), may be checked analytically. An expansion around $v=0$ is not possible, since all integrals are divergent as $y\to 0$.

To begin, consider the integral $A$, Eq. (3.1). Using Schwinger parameters [11], we can rewrite the denominator factors as exponentials, the result being:

$$A = \int_0^\infty d\alpha d\beta d\gamma \int d\!\!\bar{}\,\omega\ \exp\left\{-(\alpha+\beta)\omega_4^2 + 2\beta k_4\omega_4 - \beta k_4^2 - (\alpha+\beta+\gamma)\vec{\omega}^2 + 2\beta\vec{k}\cdot\vec{\omega} - \beta\vec{k}^2\right\}. \tag{B.1}$$

Changing variables

$$\omega_4 \to \omega_4 + \frac{\beta}{\alpha+\beta}k_4, \quad \vec{\omega}\to\vec{\omega} + \frac{\beta}{\alpha+\beta+\gamma}\vec{k} \tag{B.2}$$

completes the squares to give

$$A = \int_0^\infty d\alpha d\beta d\gamma \int d\!\!\bar{}\,\omega\ \exp\left\{-(\alpha+\beta)\omega_4^2 - \frac{\alpha\beta}{\alpha+\beta}x - (\alpha+\beta+\gamma)\vec{\omega}^2 - \frac{(\alpha+\gamma)\beta}{\alpha+\beta+\gamma}y\right\}. \tag{B.3}$$

Scaling the integration variables $\omega_4\to(\alpha+\beta)^{-1/2}\omega_4$, $\vec{\omega}\to(\alpha+\beta+\gamma)^{-1/2}\vec{\omega}$ then allows us to do the momentum integration, leaving the parametric integral

$$A = \frac{1}{(4\pi)^{2-\varepsilon}}\int_0^\infty d\alpha d\beta d\gamma (\alpha+\beta)^{-1/2}(\alpha+\beta+\gamma)^{\varepsilon-3/2}\exp\left\{-\frac{\alpha\beta}{\alpha+\beta}x - \frac{(\alpha+\gamma)\beta}{\alpha+\beta+\gamma}y\right\}. \tag{B.4}$$

By inserting the identity $1=\int_0^\infty d\lambda\delta(\lambda-\alpha-\beta-\gamma)$ and rescaling all parameters by $\lambda$, we then get

$$\begin{aligned} A &= \frac{1}{(4\pi)^{2-\varepsilon}}\int_0^1 d\alpha d\beta d\gamma\delta(1-\alpha-\beta-\gamma)(\alpha+\beta)^{-1/2}(\alpha+\beta+\gamma)^{\varepsilon-3/2}\int_0^\infty d\lambda\,\lambda^\varepsilon \exp\left\{-\lambda\frac{\alpha\beta}{\alpha+\beta}x - \lambda\frac{(\alpha+\gamma)\beta}{\alpha+\beta+\gamma}y\right\} \\ &= \frac{(x+y)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\Gamma(1+\varepsilon)\int_0^1 d\beta\int_0^{1-\beta} d\alpha\,(\alpha+\beta)^{-1/2}\left[\frac{\alpha\beta}{\alpha+\beta}\frac{z}{(1+z)} + \beta(1-\beta)\frac{1}{(1+z)}\right]^{-1-\varepsilon}. \end{aligned} \tag{B.5}$$

This last equation we denote as the parametric form of the integral. For general values of $z$, it cannot be done because of the highly nontrivial denominator factor (and clearly which is why the differential technique has been developed). However, knowing that the result is well-defined at $z = 0$, we are able to make an expansion around this point and then do the resulting parametric integrals. To second order in powers of $z$ we have:

$$
\begin{aligned}
A &\stackrel{z\to 0}{=} \frac{(x+y)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\Gamma(1+\varepsilon)\int_0^1 d\beta \int_0^{1-\beta} d\alpha\,(\alpha+\beta)^{-1/2}\times \\
&\quad \Big\{\beta^{-1-\varepsilon}(1-\beta)^{-1-\varepsilon} + (1+\varepsilon)\beta^{-\varepsilon}(1-\beta)^{-2-\varepsilon}\frac{1-\alpha-\beta}{\alpha+\beta}(z-z^2) + \frac{(1+\varepsilon)(2+\varepsilon)}{2}\beta^{1-\varepsilon}(1-\beta)^{-3-\varepsilon}\frac{(1-\alpha-\beta)^2}{(\alpha+\beta)^2}z^2 \\
&\quad + \mathcal{O}(z^3)\Big\}.
\end{aligned}
\tag{B.6}
$$

The parametric integrals can be done without difficulty and the result is

$$
A(x,y) \stackrel{z\to 0}{=} \frac{(x+y)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{-2\left(\frac{1}{\varepsilon}-\gamma\right) - 4\ln 2 + 2z - z^2 + \mathcal{O}(z^3) + \mathcal{O}(\varepsilon)\right\}
\tag{B.7}
$$

which agrees explicitly with Eq. (3.47). Actually, with hindsight and given patience in expanding the integral, it appears that this particular integral would be possible by resumming the series expansion!

Turning to the integral $A^4$, Eq. (3.2), the parametric form of the integral can be written down almost immediately given the previous case. It reads:

$$
A^4 = k_4\frac{(x+y)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\Gamma(1+\varepsilon)\int_0^1 d\beta\int_0^{1-\beta} d\alpha\,\frac{\beta}{(\alpha+\beta)^{3/2}}\left[\frac{\alpha\beta}{\alpha+\beta}\frac{z}{(1+z)} + \beta(1-\beta)\frac{1}{(1+z)}\right]^{-1-\varepsilon}.
\tag{B.8}
$$

Expanding again to second order in $z$ gives

$$
A^4(x,y) \stackrel{z\to 0}{=} k_4\frac{(x+y)^{-1-\varepsilon}}{(4\pi)^{2-\varepsilon}}\left\{4\ln 2 + \left(\frac{8}{3}\ln 2 - \frac{2}{3}\right)z + \left(-\frac{8}{15}\ln 2 - \frac{1}{15}\right)z^2 + \mathcal{O}(z^3) + \mathcal{O}(\varepsilon)\right\}
\tag{B.9}
$$

which agrees explicitly with Eq. (3.62). Although at this order it may appear reasonable to suppose that one may resum the series to recover the full function, a quick glance at the full expansion Eq. (3.62) and the solution Eq. (3.64) tells us otherwise.

As might be expected, the integral $B$, Eq. (3.3), is rather more complicated. The parametric form reads:

$$
B = \frac{(x+y)^{-\varepsilon}y^{-2}}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(2+\varepsilon)}{(1+z)^2}\int_0^1 d\alpha\int_0^{1-\alpha} d\gamma\int_0^{1-\alpha-\gamma} d\beta(\alpha+\beta)^{-1/2}\left[\frac{\alpha\beta}{\alpha+\beta}\frac{z}{1+z} + (\alpha+\gamma)(1-\alpha-\gamma)\frac{1}{1+z}\right]^{-2-\varepsilon}.
\tag{B.10}
$$

The last factor can be expanded in powers of $z$ and to second order, we get

$$
\begin{aligned}
B = \frac{(x+y)^{-\varepsilon}y^{-2}}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(2+\varepsilon)}{(1+z)^2} &\int_0^1 d\alpha\int_0^{1-\alpha} d\gamma\int_0^{1-\alpha-\gamma} d\beta(\alpha+\beta)^{-1/2}\times \\
&\Big\{(\alpha+\gamma)^{-2-\varepsilon}(1-\alpha-\gamma)^{-2-\varepsilon}\left[1+(2+\varepsilon)z+\frac{1}{2}(1+\varepsilon)(2+\varepsilon)z^2+\ldots\right] \\
&+\frac{\alpha\beta}{\alpha+\beta}(\alpha+\gamma)^{-3-\varepsilon}(1-\alpha-\gamma)^{-3-\varepsilon}\left[-(2+\varepsilon)z-(2+\varepsilon)^2z^2+\ldots\right] \\
&+\left(\frac{\alpha\beta}{\alpha+\beta}\right)^2(\alpha+\gamma)^{-4-\varepsilon}(1-\alpha-\gamma)^{-4-\varepsilon}\left[\frac{1}{2}(2+\varepsilon)(3+\varepsilon)z^2+\ldots\right]\Big\}.
\end{aligned}
\tag{B.11}
$$

The integral over $\beta$ can be done and gives

$$
\begin{aligned}
B &= \frac{(x+y)^{-\varepsilon}y^{-2}}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(2+\varepsilon)}{(1+z)^2}\int_0^1 d\alpha\int_0^{1-\alpha} d\gamma\times \\
&\quad\Big\{2\left[(1-\gamma)^{1/2}-\alpha^{1/2}\right](\alpha+\gamma)^{-2-\varepsilon}(1-\alpha-\gamma)^{-2-\varepsilon}\left[1+(2+\varepsilon)z+\frac{1}{2}(1+\varepsilon)(2+\varepsilon)z^2+\ldots\right] \\
&\quad+\left[-4\alpha^{3/2}+2\alpha^2(1-\gamma)^{-1/2}+2\alpha(1-\gamma)^{1/2}\right](\alpha+\gamma)^{-3-\varepsilon}(1-\alpha-\gamma)^{-3-\varepsilon}\left[-(2+\varepsilon)z-(2+\varepsilon)^2z^2+\ldots\right]
\end{aligned}
$$

$$+\left[2\alpha^2(1-\gamma)^{1/2}+4\alpha^3(1-\gamma)^{-1/2}-\frac{2}{3}\alpha^4(1-\gamma)^{-3/2}-\frac{16}{3}\alpha^{5/2}\right]\times$$
$$(\alpha+\gamma)^{-4-\varepsilon}(1-\alpha-\gamma)^{-4-\varepsilon}\left[\frac{1}{2}(2+\varepsilon)(3+\varepsilon)z^2+\ldots\right]\Bigg\}. \tag{B.12}$$

Now, in order to do the last pair of parametric integrals, we change variables with $\alpha = r(1-s)$, $\gamma = rs$ such that now

$$\begin{aligned}
B \;=\; & \frac{(x+y)^{-\varepsilon}y^{-2}}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(2+\varepsilon)}{(1+z)^2}\int_0^1 dr\,r\int_0^1 ds\,\times \\
& \Bigg\{2\left[(1-rs)^{1/2}-r^{1/2}(1-s)^{1/2}\right]r^{-2-\varepsilon}(1-r)^{-2-\varepsilon}\left[1+(2+\varepsilon)z+\frac{1}{2}(1+\varepsilon)(2+\varepsilon)z^2+\ldots\right] \\
& +\left[-4r^{3/2}(1-s)^{3/2}+2r^2(1-s)^2(1-rs)^{-1/2}+2r(1-s)(1-rs)^{1/2}\right]\times \\
& r^{-3-\varepsilon}(1-r)^{-3-\varepsilon}\left[-(2+\varepsilon)z-(2+\varepsilon)^2z^2+\ldots\right] \\
& +\left[2r^2(1-s)^2(1-rs)^{1/2}+4r^3(1-s)^3(1-rs)^{-1/2}-\frac{2}{3}r^4(1-s)^4(1-rs)^{-3/2}-\frac{16}{3}r^{5/2}(1-s)^{5/2}\right]\times \\
& r^{-4-\varepsilon}(1-r)^{-4-\varepsilon}\left[\frac{1}{2}(2+\varepsilon)(3+\varepsilon)z^2+\ldots\right]\Bigg\}.
\end{aligned} \tag{B.13}$$

The two-dimensional integral in $\alpha$ and $\gamma$ is now separated into two parts which can be done in turn. The integral over $s$ yields powers of $r$ and $(1-r)$ and the integral over $r$ subsequently leads to the familiar combinations of gamma functions. Expanding in $\varepsilon$ and completing the expansion of $z$ by including the prefactor $1/(1+z)^2$ we finally arrive at

$$B=\frac{(x+y)^{-\varepsilon}y^{-2}}{(4\pi)^{2-\varepsilon}}\left\{-4\left(\frac{1}{\varepsilon}-\gamma\right)\left[1-z+z^2\right]+8\ln 2\left[-\frac{1}{3}+\frac{3}{5}z-\frac{5}{7}z^2\right]-\frac{16}{3}+\frac{28}{5}z-\frac{46}{7}z^2+\mathcal{O}(z^3)+\mathcal{O}(\varepsilon)\right\} \tag{B.14}$$

which agrees explicitly with Eq. (3.79).


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